Extended Abstracts

The 1st Seminar on Control and Optimization

Ferdowsi University of Mashhad, Iran
11-12th October 2017

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Welcome

We are pleased to organize the 1st Seminar on Control and Optimization during 11-12 October 2017 in Iran. The seminar provides a forum for mathematicians worldwide and scholar students to present their latest results about all aspects of control theory and optimization such as mathematics control, control in biology, fuzzy control, stochastic control, project control, linear and nonlinear optimization, neural network and related topics, and discuss their recent researches with each other. The organizing committee of the seminar warmly welcomes the participants to Mashhad, hoping that their stay in Mashhad will be happy and fruitful. About 120 participants have taken part in this seminar. We have made every effort to make the seminar as worthwhile as possible. We wish to express our thanks to all whose help has made this gathering possible. In particular, we would like to express our gratitude to the administration of Ferdowsi University of Mashhad, the Iranian Mathematical Society, Academy of Sciences of the Islamic Republic of Iran, Center of Excellence in Analysis on Algebraic Structures, Center of Excellence on Modelling and Control Systems. The organizing committee would like to thank the many people who co-operated to make the seminar a success and this proceeding possible.

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Short Presentations

The Extended Abstracts of
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RANKING HESITANT FUZZY SETS USING THEIR HORIZONTAL REPRESENTATION

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Abstract. This paper presents a novel method for ranking hesitant fuzzy sets (HFSs) based on transforming HFSs into fuzzy sets (FSs). The idea behind the method is an interesting HFS decomposition that seems to be a missing discussion in the relevant literature and we refer to it as the horizontal representation.

1. Introduction

A large number of ranking methods for FSs have been suggested in the literature so far [2]. However, few studies focus on the ranking methods for HFSs [1]. To put forward some formulas to get the ranking order of HFSs, we are motivated to propose a new ranking method for HFSs which is intuitive in nature, computationally simple and easy to implement.

2. Transformation of HFSs into FSs

Throughout this article, we use $X = \{x_1, x_2, ..., x_N\}$ to denote the discourse set. A fuzzy set(FS) A on X is defined as $A = \{\langle x, A(x)\rangle : x \in X\}$, where $A(x)$ is the degree of membership of $x \in X$ in A. We

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* Speaker.

denote all FSs on X by $FS(X)$. Formally, any aggregation operator on a collection of n FSs $A_1, A_2, ..., A_n$ defined on X is characterized by a mapping

$$\sigma : [0,1]^n \to [0,1],$$
$$\sigma(A_1(x), A_2(x), ..., A_n(x)) := A^\sigma(x), \quad \forall x \in X,$$

where A^σ is called the aggregated fuzzy set and also $A^\sigma \in FS(X)$. For a collection of FSs $A_i (i = 1, 2..., n)$ and the weight vector of A_i, denoted by $W = (w_1, w_2, ..., w_n)$ with $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, then *the fuzzy weighted averaging (FWA) operator* is defined as:

$$FWA(A_1, A_2, ..., A_n) = \oplus_{i=1}^n (w_i A_i) = 1 - \prod_{i=1}^n (1 - A_i)^{w_i}.$$

Now, inspired by the fact that "aggregation operations on fuzzy sets are mappings by which a collection of fuzzy sets are combined in a desirable way to give rise to a single fuzzy set", we establish a transformation of HFSs into FSs. A hesitant fuzzy set (HFS) H on X is defined [1] in terms of a function $h_H(x)$ when applied to X returns a finite subset of $[0,1]$, i.e.,

$$H = \{\langle x, h_H(x)\rangle | x \in X\}, \tag{2.1}$$

where $h_H(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x \in X$ to H. We denote all HFSs on X by $HFS(X)$, and for convenience, we call $h_H(x)$ a hesitant fuzzy element (HFE) [3]. Now, we introduce a new concept of a HFS which is called a $\kappa-$level set of the HFS and expressed in the following definition:

Definition 2.1. For given $H \in HFS(X)$, we define the fuzzy set $h_H^{[\kappa]}$ as the $\kappa-$level set of HFS H where

$$h_H^{[\kappa]}(x_i) = \begin{cases} h_H^{\delta(\kappa)}(x_i) & if\ \kappa \leq l_H, \\ h_H^{\delta(l_H)}(x_i) & if\ \kappa > l_H, \end{cases} \tag{2.2}$$

where $h_H^{\delta(\kappa)}$ denotes the k-th largest element, and moreover, the number of $\kappa-$level sets of HFS H is l_H.

Proposition 2.2. *The $\kappa-$level sets of the HFS H are increasing with respect to κ, that is, if $\kappa_1 \leq \kappa_2$, then $h_H^{[\kappa_1]} \preceq h_H^{[\kappa_2]}$.*

As a corollary of Proposition 2.2, we can conclude for any $\kappa-$level set of the HFS H that

$$h_H^{[1]} \preceq h_H^{[\kappa]} \preceq h_H^{[l_H]}, \quad 1 \leq \kappa \leq l_H, \tag{2.3}$$

which implies that any $\kappa-$level sets of a HFS H is bounded from below and bounded from above. The following argument shows that there are two ways of representing a HFS.
The first one is the most commonly used representation of a HFS that we refer to as *the vertical representation*. It is based on HFEs of HFS H and expressed by

$$H = \overline{\sum}_{x_i \in X} h_H(x_i),$$

where $\overline{\sum}$ denotes union over all $x_i \in X$.
The second one that does seem to be a major missing discussion in the literature is that we refer to as *the horizontal representation*.
Before we further discuss on the second representation, it needs to consider the relation between HFEs and $\kappa-$level sets of HFS H, where for any fixed $x_i \in X$

$$h_H(x_i) = \overline{\sum}_{1 \leq \kappa \leq l_H} h_H^{[\kappa]}(x_i), \tag{2.4}$$

where $\overline{\sum}$ denotes union over all $1 \leq \kappa \leq l_H$.
Now, we introduce the second representation as follows:

Theorem 2.3. *(Horizontal representation) A HFS H is represented by the union of all its $\kappa-$level sets, that is,*

$$H = \overline{\sum}_{1 \leq \kappa \leq l_H} h_H^{[\kappa]}, \tag{2.5}$$

where $\overline{\sum}$ denotes union over all $1 \leq \kappa \leq l_H$.

The horizontal representation would be an interesting HFS decomposition as we have not discovered it in the relevant literature. As will be seen later, the transforming method of HFSs into FSs is established on the basis of this novel representation. Taking a look at the concept of $\kappa-$level sets introduced in Definition 2.1, we define comparable HFSs as follows:

Definition 2.4. The two HFSs H_1 and H_2 are said to be comparable and denoted by $H_1 \preceq H_2$ (or $H_1 \succeq H_2$) if and only if $h_{H_1}^{[\kappa]} \preceq h_{H_2}^{[\kappa]}$ (or $h_{H_1}^{[\kappa]} \succeq h_{H_2}^{[\kappa]}$) for any $1 \leq \kappa \leq \max\{l_{H_1}, l_{H_2}\}$. In the case that $l_{H_1} < l_{H_2}$, to operate correctly, put $h_{H_1}^{[\kappa]} = h_{H_1}^{[l_{H_1}]}$ for all $\kappa > l_{H_1}$.

Definition 2.5. Let $H \in HFS(X)$ and $\sigma : [0,1]^{l_H} \to [0,1]$ be an aggregation operator given by (Agg1). Then, corresponding to HFS H we define the following FS

$$H^\sigma := \sigma(h_H^{[1]}, h_H^{[2]}, ..., h_H^{[l_H]}), \tag{2.6}$$

where $h_H^{[\kappa]}$'s, $(1 \leq \kappa \leq l_H)$ are $\kappa-$level sets of HFS H.

3. New ranking method for HFSs

Proposition 3.1. *Let $H_1, H_2 \in HFS(X)$ with $l_H := l_{H_1} = l_{H_2}$, and $FWA : [0,1]^{l_H} \to [0,1]$ be the aggregation operator given by (Agg1). Corresponding to HFSs H_1,H_2 we define*

$$H_1^{FWA} := FWA(h_{H_1}^{[1]}, h_{H_1}^{[2]}, ..., h_{H_1}^{[l_H]}), \tag{3.1}$$

$$H_2^{FWA} := FWA(h_{H_2}^{[1]}, h_{H_2}^{[2]}, ..., h_{H_2}^{[l_H]}), \tag{3.2}$$

where $h_{H_i}^{[\kappa]}$'s, $(1 \leq \kappa \leq l_H)$ are $\kappa-$level sets of HFSs H_i, $i = 1, 2$. Then, there is a correspondence between the ranking of HFSs H_1, H_2 and the ranking of their aggregated FSs H_1^{FWA}, H_2^{FWA} that can be stated as

$$\begin{aligned} H_1 \prec H_2 \quad iff \quad H_1^{FWA} \prec H_2^{FWA}, \\ H_1 \succ H_2 \quad iff \quad H_1^{FWA} \succ H_2^{FWA}, \\ H_1 \approx H_2 \quad iff \quad H_1^{FWA} \approx H_2^{FWA}. \end{aligned}$$

Algorithm 3.2. (The new method for ranking HFSs)
Let $\{H_1, H_2, ..., H_m\}$ be a collection of m HFSs on $X = \{x_1, x_2, ..., x_N\}$. Then, use the following steps to find the ordering of given HFSs.
Step 1. For any HFS H_i $(i = 1, 2, ..., m)$, we construct the corresponding $\kappa-$level sets $h_{H_i}^{[\kappa]}$'s $(\kappa = 1, 2, ..., l_{H_i})$ using Definition 2.1.
Step 2. Using Definition 2.5 in which the aggregation operator σ is that given by (Agg1), we construct the aggregated FS

$$H_i^{FWA} := FWA(h_{H_i}^{[1]}, h_{H_i}^{[2]}, ..., h_{H_i}^{[l_{H_i}]}),$$

corresponding to each HFS H_i $(i = 1, 2, ..., m)$.
Step 3. Known by Proposition 3.1, the ordering of HFSs $\{H_1, H_2, ..., H_m\}$ can be achieved according to the ordering of their aggregated FSs $\{H_1^{FWA}, H_2^{FWA}, ..., H_m^{FWA}\}$.

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A NEW METHOD TO FIND THE APPROXIMATE OPTIMAL CONTROL PROBLEM GOVERNED BY NONLINEAR VOLTERRA INTEGRAL EQUATION

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Abstract. In this paper, a new method for solving optimal control problems governed by nonlinear Volterra integral equations is presented. For this purpose, the problem is first converted to discrete form. It is then considered as a quasi assignment problem and an iterative method is applied to find approximate solution for discretized form of the integral equation. Then by using evolutionary algorithms, approximate solution of optimal control problems is obtained.

1. Introduction

Evolutionary algorithms (EAs) as well as optimization are two prominent fields of research in applied science and engineering. Recently, evolutionary and heuristic algorithms have been raised as powerful tools in solving optimal control problems, [1, 2, 3]. Combination of these approaches and a usual numerical approach of solving ODE's with dicretization of control space leads to efficient numerical scheme for detecting approximate optimal control and state functions in classical optimal control problems. In this

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paper, we focus on the formulation of a class of optimal control problems governed by nonlinear Volterra integral equations as follows

$$\text{Minimize } J(x,u) = \int_0^T \zeta(t, x(t), u(t))dt, \tag{1.1}$$

where control function $u(.)$ and corresponding state $x(.)$ are subjected to

$$x(t) = y(t) + \int_0^t \kappa(t, s, x(s), u(s))ds, \text{ a.e. on } [0,T]. \tag{1.2}$$

Here $\zeta \in C([0,T]\times\mathbb{R}\times\mathbb{R})$ and $\kappa, \kappa_x(=\frac{\partial\kappa}{\partial x}) \in C([0,T]\times[0,T]\times\mathbb{R}\times\mathbb{R})$.

2. Discretization of Control Space

In this section we present a control discretization based method by equidistance partition of $[0,T]$ as $\triangle_n = \{0 = t_0, t_1, \cdots, t_{n-1}, t_n = T\}$ with discretization parameter $h = t_{i+1} - t_i$, $i = 0, 1, \cdots, n-1$. The time interval is divided to n sub-interval $[t_0 = 0, t_1]$, $[t_1, t_2]$, $\cdots$, $[t_{n-1}, t_n = T]$. On the other hand, the set of control values is divided to constants $u_1, u_2, \cdots, u_m$. In this way, the time-control space is discretized if the control function assumes to be constant at each time sub-interval. Using the characteristic function

$$\chi_{[t_{k-1},t_k)}(t) = \begin{cases} 1 & t \in [t_{k-1}, t_k), \\ 0 & \text{otherwise}, \end{cases}$$

the control function may be presented as $u(t) = \sum_{k=1}^{N} u_k \chi_{[t_{k-1},t_k]}(t)$. Trivially, the corresponding trajectory should be in discretized form. Thus a discretized form of the problem (1.1)-(1.2) should be considered such that its solution be converged to the solution of the original problem.

Now, if (x,u) be an admissible pair, then for the partition $\triangle_n$ on $[0,T]$, we have

$$x(t_i) = y(t_i) + \int_0^{t_i} k(t_i, s, x(s), u(s))ds, \ i = 0, 1, \cdots, n. \tag{2.1}$$

In (2.1), the term integral can be estimated by a numerical method of integration, e.g. one of Newton-Cotes methods. Therefore, by taking equidistance partition $\triangle_n$, as above with $h = t_{i+1} - t_i$, $i = 0, 1, \cdots, n-1$ and also the weights w_{i_j}, $j = 0, 1, \cdots, i$, equality (2.1) can be written as,

$$x_i = y_i + \sum_{j=0}^{i} w_{i_j} k(t_i, s_j, x_j, u_j) + O(h^\nu), \ i = 0, 1, \cdots, n, \tag{2.2}$$

where $x_i = x(t_i)$, $y_i = y(t_i)$, $i = 0, 1, \cdots, n$, and ν depends upon the used method of Newton-Cotes for estimating of the integral in (2.1). The same partition and weights can be used to convert the objective function (1.1) to the following form

$$J(x,u) = \sum_{j=0}^{n} w_j \zeta(t_j, x_j, u_j) + O(h^\nu). \tag{2.3}$$

For partition $\triangle_n$, by neglecting the truncation error of (2.2) and (2.3), the following nonlinear optimization problem may be considered

$$\text{Minimize} \qquad J_{\triangle_n} = \sum_{j=0}^{n} w_j \zeta(t_j, \xi_j, \upsilon_j), \tag{2.4}$$

$$\text{subject to: } \xi_i = y_i + \sum_{j=0}^{i} w_{i_j} k(t_i, s_j, \xi_j, \upsilon_j),\ i = 0, 1, \cdots, n, \tag{2.5}$$

3. Convergence

The solution of nonlinear programming (2.4)-(2.5) approximates the original problem by minimizing $J(x, u)$ over the subset $\mathcal{P}_n$ of $\mathcal{P}$ consist of all piece-wise linear function $x(.)$ and $u(.)$ with nodes at $\xi_0, \xi_1, \cdots \xi_n$ and $\upsilon_0, \upsilon_1, \cdots, \upsilon_n$ satisfying (2.4). Our first aim is to show that $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \cdots$ in an embedding fashion.

Lemma 3.1. *There exists an embedding that maps $\mathcal{P}_n$ to a subset of $\mathcal{P}_{n+1}$ for all $n = 1, 2, \cdots$.*

The above lemma has an important result in decreasing behavior of the optimal value of the objective function which leads to the following theorem.

Theorem 3.2. *If $\mu_n = \inf_{\mathcal{P}_n} J_{\triangle_n}$ for $n = 1, 2, \cdots$, and $\mu^* = \inf_{\mathcal{P}} J(x, u)$, then $\lim_{n\to\infty} \mu_n = \mu^*$.*

4. Combination Approach

For an successive iterative scheme of solving nonlinear Volterra integral equations, we apply a successive substitution, similar to Gauss-Seidel method of solving linear equations systems, and thereby define an iterative process leading to the sequence of vectors $\{\xi^{(k)}\}$, where the components of the vectors satisfy the iteration formula,

$$\xi_i^{(k+1)} = y_i + \sum_{j=0}^{i} w_{i_j} \kappa(t_i, s_j, \xi_j^{(k)}, \upsilon_j),\ i = 0, 1, \cdots, n,\ k = 0, 1, \cdots \tag{4.1}$$

where we consider a partition $\triangle_n$ on the time interval $[0, T]$ and a discretization of the control space on basis of this partition.

Proposition 4.1. *Suppose,*
(i) $\kappa(t, s, \xi(s), \upsilon(s)) \in C([0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R})$,
(ii) $\kappa_\xi(t, s, \xi(s), \upsilon(s))$ *exists on* $[0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R}$ *and* $\gamma < \frac{1}{T}$, *where*

$$\gamma = \sup_{s,t\in[0,T]} |\kappa_\xi(t, s, \xi(s), \upsilon(s))|.$$

Then

$$\|x^* - \xi^*\|_\infty \leq \frac{|O(h^\nu)|}{1 - T\gamma} \tag{4.2}$$

where $x^* = (x_0^*, x_1^*, \cdots, x_n^*)^T$ and $\xi^* = (\xi_0^*, \xi_1^*, \cdots, \xi_n^*)^T$, are the exact solutions of nonlinear systems (2.2) and (2.5), respectively.

Corollary 4.2. *$\|x^* - \xi^*\|_\infty$ vanishes when $h \to 0$.*

Theorem 4.3. *Considering assumptions of Proposition 4.1, the produced sequence $\{\xi^{(k)}\}$ from the iteration process (4.1) tends to the exact solution of (2.5), say ξ^*, for any arbitrary initial vector $\xi^{(0)}$.*

Now an evolutionary algorithm, as genetic algorithm, can be applied by considering the performance index (2.4) for an approximate admissible pair (ξ, υ). Assuming $\xi^{(k)}$ be the vector which is obtained in the kth iteration, a stopping criteria may be considered as follows:

$$\frac{\|\xi^{(k+1)} - \xi^{(k)}\|}{\|\xi^{(k)}\|} < \epsilon, \tag{4.3}$$

for a prescribed small positive number ϵ that should be chosen according to the accuracy desired, where $\|\cdot\|$ is a norm on vectors. We have applied three evolutionary algorithms, i.e. genetic algorithm (GA), particle swarm optimization (PSO) and invasive weed optimization (IWO).

Example 4.4. Consider the following optimal control

$$\text{Minimize } \int_0^1 (x(t) - t)^2 + (u(t) - t^2)^2 dt,$$

subject to:

$$x(t) = y(t) + \int_0^t x(s)(u(s) + ts)ds,$$

where, $y(t) = t - \frac{7}{12}t^4$. The results of applying the proposed algorithm with the number of iterations=100, population size=10, are illustrated in Fig.1, where the approximate optimal trajectories and controls are compared with the exact ones, respectively.

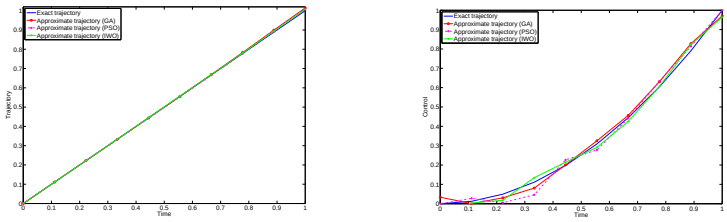

FIGURE 1. The exact and approximate solutions in Example 4.4.

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A CHEBYSHEV SPECTRAL METHOD TO FIND THE EXTREMUM OF THE FUNCTIONAL $J[u] = \int_{-1}^{1}\int_{-1}^{1}\left[(u_x)^2 + (u_y)^2 - 2u\right]\,dx\,dy$

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Abstract. In this paper, we present a Chebyshev spectral method for finding the solution of the variational problem which is stated as: Minimize $J[u] = \int_{-1}^{1}\int_{-1}^{1}\left[(u_x)^2 + (u_y)^2 - 2u\right]\,dx\,dy$, subject to the boundary conditions $u(x,\pm 1) = u(\pm 1, y) = 0$, $-1 \le x \le 1$, $-1 \le y \le 1$. The method is based upon constructing the interpolating polynomial for the function $u(x,y)$, using Chebyshev nodes, to approximate the solution of the variational problem. Using the Clenshaw integration rule, the functional of the variational problem is discretized to an algebraic expression. Therefore, the variational problem is reduced to a nonlinear programming problem. The numerical results demonstrate the convergence of the proposed method.

1. Introduction

In this talk, we consider the following variational problem: Find the function $u = u(x,y)$ that satisfies the boundary conditions

$$u(x,\pm 1) = u(\pm 1, y) = 0,\ -1 \le x \le 1,\ -1 \le y \le 1, \tag{1.1}$$

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* Speaker.

and minimizes the functional

$$J[u] = \int_{-1}^{1}\int_{-1}^{1} \left[(u_x)^2 + (u_y)^2 - 2u\right]\, dx\, dy. \tag{1.2}$$

Problems in which it is required to determine a function which maximizes or minimizes a certain functional are called variational problems. The numerical methods such as the well-known Ritz and Galerkin methods have been developed to solve variational problems [1, 2]. The Ritz and Galerkin methods are direct methods which convert the variational problem to a mathematical programming problem. For instance, the method in [4] requires that the performance index and the system differential equations be expanded around nominal trajectories. Therefore, the original optimal control problem can be solved by solving a sequence of linear-quadratic optimal control problems. Then, each of these obtained problems is converted into a quadratic programming problem using Chebyshev polynomials to parameterize the state variables.

The purpose of this talk is to present an alternative approach. Here, we introduce a Chebyshev spectral method for finding the solution of the variational problem (1.2) with the boundary conditions defined in Eq. (1.1). The approach is a spectral method in which we construct the interpolating polynomial using Chebyshev nodes to approximate the function $u(x, y)$. The partial derivatives $u_x(x, y)$ and $u_y(x, y)$ are approximated by analytic derivatives of the corresponding interpolating polynomial. The functional (1.2) is discretized using the Clenshaw integration rule. Therefore, the variational problem is reduced to a nonlinear programming problem to which existing well-developed algorithms could be applied.

It is well-known that in order for the function $u = u(x, y)$ to be a solution of the variational problem (1.2), u must be an extremal, i.e., a solution of the Euler's equation [3]. It is readily verified that the Euler's equation corresponding to the variational problem (1.2) has the following form, known as Poisson's equation:

$$-(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) = 1. \tag{1.3}$$

Therefore, by applying the present method to approximate the solution of the variational problem stated in this talk, we obtain an approximate solution to the Poisson's equation (1.3) subject to the boundary conditions defined in Eq. (1.1).

2. Proposed Method

In this approach, we expand the function $u(x, y)$ of two variables x and y, in terms of Lagrange polynomials of degree m, as

$$u_m(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{m} u_{ij}\, L_i(x)\, L_j(y), \tag{2.1}$$

where $U = [u_{ij}]$ is the $(m + 1) \times (m + 1)$ matrix of unknown coefficients $u_{ij} = u(x_i, y_j)$, $i, j = 0, 1, \ldots, m$. Here, we let $\{x_0, x_1, \ldots, x_m\} = \{y_0, y_1, \ldots, y_m\}$ be the set of Chebyshev nodes defined by $x_j = \cos(\frac{\pi j}{m})$, $j = 0, 1, \ldots, m$ and suppose $L_i(x)$ and $L_j(y)$ are the Lagrange polynomials of degree m corresponding to x_i and y_j, respectively.

Now, by substituting $u(x, y)$ by $u_m(x, y)$ in the functional (1.2) and using the Clenshaw-Curtis quadrature formula, we obtain the following discretization of the functional (1.2):

$$\begin{aligned} &J_m[u_m] \\ &= \sum_{i=0}^{m} \sum_{j=0}^{m} w_i w_j \{(\frac{\partial u_m}{\partial x}(x_i, y_j))^2 + (\frac{\partial u_m}{\partial y}(x_i, y_j))^2 - 2u_{ij}\}, \end{aligned} \tag{2.2}$$

where w_j, $j = 0, 1, \ldots, m$, are the weights of the Clenshaw-Curtis quadrature formula. In order to compute the value of $\frac{\partial u_m}{\partial x}$ at the node $(x, y) = (x_i, y_j)$, we firstly differentiate the interpolating polynomial $u_m(x, y)$ with respect to x and then we set $(x, y) = (x_i, y_j)$. By this approach, we obtain

$$\frac{\partial u_m}{\partial x}(x_i, y_j) = \sum_{k=0}^{m} d_{ik} u_{kj},$$

where $D = [d_{ij}]$ is the Chebyshev differentiation matrix [5]. Similarly, by differentiating the interpolating polynomial $u_m(x, y)$ with respect to y, the value of $\frac{\partial u_m}{\partial y}$ at the node (x_i, y_j) is obtained by the summation

$$\frac{\partial u_m}{\partial y}(x_i, y_j) = \sum_{k=0}^{m} d_{jk} u_{ik}.$$

Therefore, by applying the proposed method, the variational problem (1.2) with the boundary conditions defined in Eq. (1.1) is reduced to a parameter optimization problem which can be stated as follows: Find $U = [u_{ij}]$, the $(m + 1) \times (m + 1)$ matrix of unknown coefficients $u_{ij} = u(x_i, y_j)$, $i, j = 0, 1, \ldots, m$, that minimizes the expression given in Eq. (2.2). Note that, using the boundary conditions defined in Eq. (1.1), $u_{0j} = u_{mj} = u_{i0} = u_{im} = 0$, $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, m$, are given. By solving the obtained nonlinear programming problem for

the unknowns u_{ij}, $i, j = 0, 1, \ldots, m$, the approximate solution of the variational problem (1.2) with the boundary conditions defined in Eq. (1.1) is obtained using Eq. (2.1).

3. The numerical results

In Table 1, we reported the values of the maximum absolute error $E = \{|u(x, y) - u^*(x, y)|, -1 \leq x, y \leq 1\}$, where $u(x, y)$ and $u^*(x, y)$ denote the approximate solution obtained by the proposed method and the exact solution, respectively. In this table, we also listed the approximate values of $J[u]$ which are obtained by applying the proposed method. Note that the extremum value of the functional $J[u]$ is $J[u^*] = -0.562308$.

Table 1. Computational results for the maximum absolute error E and approximate values of $J[u]$.

	$J[u]$	E
$m = 3$	-0.592592	3.9×10^{-2}
$m = 5$	-0.563424	3.1×10^{-3}
$m = 7$	-0.562371	7.5×10^{-4}
$m = 9$	-0.562316	2.7×10^{-4}
$m = 11$	-0.562310	9.7×10^{-5}
$m = 13$	-0.562308	3.6×10^{-5}

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AN EFFICIENT METHOD BASED ON MÜNTZ-LEGENDRE POLYNOMIALS FOR SOLVING DELAY FRACTIONAL OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, a numerical method is applied for solving delay fractional optimal control problems (DFOCPs). At the first step, using Padé approximation, the delay problem is transformed to a non-delay problem. Next, using the operational matrix of the fractional derivative of Müntz polynomials and pseudospectral (PS) method, fractional optimal control problem (FOCP) is reduced into a nonlinear programming problem. A numerical example is given to illustrate the effectiveness of the proposed scheme.

1. Introduction

In this paper, we are interested in delay fractional optimal control problem

$$J = \int_0^1 G(t, x(t), x(t-\sigma), u(t), u(t-\tau))dt, \tag{1.1}$$

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* Speaker.

$$\text{subject to } {}^{C}D^{\alpha}x(t) = F(t, x(t), x(t-\sigma), u(t), u(t-\tau)), \quad t \in [0,1], \tag{1.2}$$

$$g(t, x(t), u(t)) \leqslant 0, \quad t \in [0,1], \tag{1.3}$$

$$x(t) = \phi(t), \quad t \in [-\sigma, 0], \tag{1.4}$$

$$u(t) = \psi(t), \quad t \in [-\tau, 0], \tag{1.5}$$

where ${}^{C}D^{\alpha}$ is Caputo fractional derivative which is defined as follows

$${}_{0}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \ \alpha \in (n-1, n), \ n \in \mathbb{N}. \tag{1.6}$$

2. Transformation of delay to non-delay problems

2.1. **Padé approimantion for DFOCPs.** Two-sided Laplace transform of $f(t)$ is defined as follows [3]

$$\mathcal{B}(f(t)) = F(s) \triangleq \int_{-\infty}^{\infty} e^{-st} x(t) dt,$$

To remove variables with a time-delayed argument in (1.1)-(1.2), we first define $y(t) \triangleq x(t-\sigma)$ and $\mathcal{B}(y(t)) = Y(s)$. The two-sided Laplace transforms $Y(s)$ and $X(s)$ are then related by (see [3])

$$Y(s) = e^{-s\sigma} X(s). \tag{2.1}$$

Using the first-order Padé approximation, equation (2.1) is approximated by

$$\begin{cases} Y(s) = \frac{\frac{2}{\sigma} - s}{\frac{2}{\sigma} + s} X(s), \\ (\frac{2}{\sigma} + s) Y(s) = (\frac{2}{\sigma} - s) X(s). \end{cases} \tag{2.2}$$

If we now perform an inverse two-side Laplace transformation on the last equation (2.2), we have

$$\dot{y}(t) = \frac{2}{\sigma}(x(t) - y(t)) - \dot{x}(t).$$

We can obtain similar relations for delay in control. The time-delayed problem (1.1)-(1.5) is thus transformed to non-delayed problem that can be solved by many algorithms for the FOCPs.

3. Numerical treatment of the FOCP

In this section, we propose a numerical scheme based on PS method to solve the FOCP.

3.1. **Müntz–Legendre polynomials.** The MLPs on the interval $[0,1]$ are represented by the formula

$$L_{n,\alpha}(t)=\sum_{k=0}^{n}C_{n,k}t^{k\alpha},\ C_{n,k}=\frac{(-1)^{n+k}}{\alpha^n k!(n-k)!}\prod_{v=0}^{n-1}((k+v)\alpha+1). \quad (3.1)$$

3.2. **Approximation of FOCP.** In our approach, first we obtain operational matrix of fractional derivative for MLPs. Then we use the quadrature rule to approximate the integral involved in the cost function. In the discretization of the FDE, we will use the shifted Legendre-Gauss nodes (i.e. roots of $P_{N+1}(t)=0$). We now turn to define shifted Legendre-Gauss nodes and corresponding quadrature weights. We denote by $x_k, k=0,1,2,\cdots,N$, the standard Legendre-Gauss nodes on the interval $(-1,1)$, which are the zeros of $L_{N+1}(x)$. The shifted Legendre-Gauss nodes on the interval $[0,1]$ denoted by $\eta_k, k=0,\cdots,N$. Clearly, $\eta_k=\frac{x_k}{2}+\frac{1}{2}$. The corresponding quadrature weights are $\hat{w}_k=\frac{w_k}{2}$.
We now assume that, the solutions of the final problem can be approximated by the MLPs as

$$\begin{cases} x(t)\approx x_N(t)=\mathcal{X}^T\mathcal{L}(t)=\sum_{i=0}^{N}X_iL_{i,\alpha}(t),\\ y(t)\approx y_N(t)=\mathcal{Y}^T\mathcal{L}(t)=\sum_{i=0}^{N}Y_iL_{i,\alpha}(t),\\ u(t)\approx u_N(t)=\mathcal{U}^T\mathcal{L}(t)=\sum_{i=0}^{N}U_iL_{i,\alpha}(t),\\ v(t)\approx v_N(t)=\mathcal{V}^T\mathcal{L}(t)=\sum_{i=0}^{N}V_iL_{i,\alpha}(t), \end{cases} \quad (3.2)$$

where $X_i,Y_i,U_i,V_i, i=0,1,2,\cdots,N$, are the unknown MLPs coefficients to be determined and $L_{i,\alpha}(t)s$ are MLPs of order α. Thus, considering the above equations at ShLG collocation points, the obtained FOCP will be transform to a NLP in a structured form. Since the gradient of the objective function is available we use sequential quadratic programming (SQP). We implement this method using Maple 18 software.

4. Numerical example

Example 4.1. Consider the following DFOCP

$$\text{minimize } J=\frac{1}{2}\int_0^2(x^2(t)+u^2(t))dt, \quad (4.1)$$

$$\text{subject to } {}_0^C D_t^{\alpha}x(t)=x(t-1)+u(t), \quad (4.2)$$

$$x(t)=1,\ t\in[-1,0]. \quad (4.3)$$

This problem is solved using the suggested method. The values of the cost functional J for $\alpha=1,0.9,0.8$ and $N=20$ are reported in

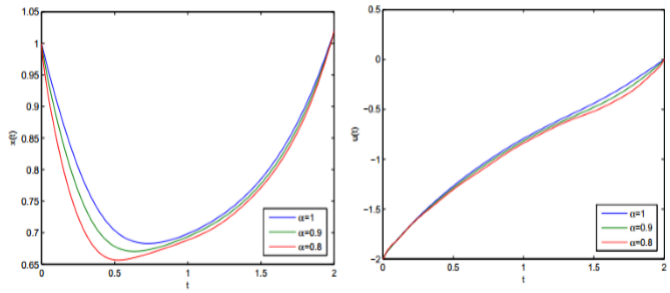

FIGURE 1. Approximate solutions of $x(.)$ and $u(.)$ for $\alpha = 1, 0.9, 0.8$ and $N = 20$

Table 1. The state and control functions for $N = 20$ are also shown in Figure 1.

TABLE 1. Approximate values of cost function J in Example 5.1.

Method	Value of α	Value of J
The presented method	$\alpha = 1$	1.647453
$N = 20$	$\alpha = 0.9$	1.657988
	$\alpha = 0.8$	1.668994
Walsh functions [2]	$\alpha = 1$	1.6497
$N = 100$		
Hybrid function [1]	$\alpha = 1$	1.647874
$N = 3, M = 6$		

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AN APPROXIMATE SOLUTION FOR OPTIMAL CONTROL PROBLEM OF BURGERS EQUATION

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Abstract. In this paper, the Chebyshev pseudo-spectral method is applied for solving optimal control of Burgers equation. In this method, an interpolating polynomial is utilized to approximate the optimal solution which satisfies some optimality conditions. We show that the proposed method has high accuracy and high convergence rate.

1. Introduction

Optimal control of viscous Burgers equation is one of the most important PDEs constraint optimization which is taken into consideration and several papers have been presented in its numerical solution.
In this paper, we apply an indirect chebyshev pseudo-spectral (CPS) method for solving optimal control of Burgers equation. Here, we utilize the CPS method to solve the optimality equations. By numerical example, we see that the CPS method is more effective than method given by sabeh et al.[1] and we can achive the better results for the solution of optimal control problem of Burgers equation.

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Key words and phrases. Burgers equation, Optimal control, Chebyshev pseudo-spectral method, Chebyshev-Gauss-Lobatto nodes.

* Speaker.

2. Optimal control problem of Burgers equation

The distributed optimal control problem for the Burgers equation can be stated as follows

$$\text{Minimize} \quad J[y,u] = \frac{1}{2}\int_0^T \int_0^1 (y(t,x) - z(t,x))^2 dxdt + \frac{\alpha}{2}\int_0^T \int_0^1 u^2(t,x)dxdt \tag{2.1}$$

$$s.t \begin{cases} y_t(t,x) + y(t,x)y_x(t,x) - \nu y_{xx}(t,x) = \Phi(u), (t,x) \in Q = [0,T] \times [0,1] \\ y(t,0) = y(t,1) = 0, \quad t \in \Sigma = [0,T] \\ y(0,x) = y_0(x), \quad x \in \Omega = [0,1] \end{cases} \tag{2.2}$$

where $y(.,.)$ is the state variable, $u(.,.)$ is the control variable, $\alpha > 0$ is the regularization parameter, $\nu > 0$ denotes the viscosity parameter and Φ is a given function.

At the first, optimality conditions for problem (2.1)-(2.2) are given, then we indirectly develop the CPS method to achieve an approximate optimal solution.

The first-order optimality conditions for the problem (2.1)-(2.2) are given as follows (see [2, 3])

$$\begin{cases} y_t - \nu y_{xx} + yy_x = \Phi(u), & (t,x) \in Q, \\ p_t + \nu p_{xx} + yp_x = y_d - y, & (t,x) \in Q, \\ y(t,0) = y(t,1) = 0, & t \in \Sigma, \\ y(0,x) = y_0, & x \in \Omega, \\ p(t,0) = p(t,1) = 0, & t \in \Sigma, \\ p(T,x) = 0, & x \in \Omega, \\ \alpha u + p = 0, & (t,x) \in Q. \end{cases} \tag{2.3}$$

From the last equation of system (2.3), we have

$$u = -\frac{1}{\alpha}p. \tag{2.4}$$

Now, we utilize the CPS method to solve optimality conditions (2.3) and obtain an approximate optimal solution for the optimal control problem (2.1)-(2.2).

To use the CPS method the variables of system (2.3) must be transformed to interval $[-1,1]$ by the following linear transformations

$$t = \frac{T}{2}\bar{t} + \frac{T}{2} \quad x = \frac{1}{2}\bar{x} + \frac{1}{2}; \quad x \in [0,1], \quad t \in [0,T], \quad \bar{x}, \bar{t} \in [-1,1]. \tag{2.5}$$

By (2.4) and (2.5), the optimality conditions (2.3) can be written as follows

$$\begin{cases} \frac{2}{T}Y_{\bar{t}} - 4\nu Y_{\bar{x}\bar{x}} + 2YY_{\bar{x}} = \Phi(-\frac{1}{\alpha}P), & in\ [-1,1]\times[-1,1], \\ \frac{2}{T}P_{\bar{t}} + 4\nu P_{\bar{x}\bar{x}} + 2YP_{\bar{x}} = Y_d - Y, & in\ [-1,1]\times[-1,1], \\ Y(\bar{t},0) = Y(\bar{t},1) = 0, & in\ [-1,1], \\ P(\bar{t},0) = P(\bar{t},1) = 0, & in\ [-1,1], \\ Y(0,\bar{x}) = Y_0(\bar{x}), & in\ [-1,1], \\ P(T,\bar{x}) = 0, & in\ [-1,1]. \end{cases} \tag{2.6}$$

Now, to approximate the optimal solution, we utilize the following polynomial interpolations

$$\begin{cases} Y^N(\bar{t},\bar{x}) = \sum_{i=0}^{N}\sum_{j=0}^{N}\bar{a}_{ij}L_i(\bar{t})L_j(\bar{x}), \\ P^N(\bar{t},\bar{x}) = \sum_{i=0}^{N}\sum_{j=0}^{N}\bar{b}_{ij}L_i(\bar{t})L_j(\bar{x}) \end{cases} \tag{2.7}$$

To express the derivatives we can use the matrix multiplication D where

$$D_{kj} = L_j'(\bar{t}_k) = \begin{cases} \frac{\mu_k}{\mu_j}(-1)^{k+j}\frac{1}{\bar{t}_k - \bar{t}_j}, & if\ j \neq k, \\ -\frac{\bar{t}_k}{2-2\bar{t}_k^2}, & if\ 0 \leq j = k \leq N-1, \\ -\frac{2N^2+1}{6}, & if\ j = k = 0, \\ \frac{2N^2+1}{6}, & if\ j = k = N. \end{cases} \tag{2.8}$$

Now, by relations (2.7) and (2.8), conditions (2.6) can be written as the following discrete form

$$\begin{cases} \frac{2}{T}\sum_{i=0}^{N}\bar{a}_{ik}D_{pi} - 4\,\nu\sum_{i=0}^{N}\sum_{j=0}^{N}\bar{a}_{pi}D_{ji}D_{kj} + 2\,\bar{a}_{pk}\sum_{j=0}^{N}\bar{a}_{pj}D_{kj} = \Phi(-\frac{1}{\alpha}\bar{b}_{pk}), \\ \frac{2}{T}\sum_{i=0}^{N}\bar{b}_{ik}D_{pi} + 4\,\nu\sum_{i=0}^{N}\sum_{j=0}^{N}\bar{b}_{pi}D_{ji}D_{kj} + 2\,\bar{a}_{pk}\sum_{j=0}^{N}\bar{b}_{pj}D_{kj} = Y_d(\bar{t}_p,\bar{x}_k) - \bar{a}_{pk}, \\ \bar{a}_{p0} = \bar{a}_{pN} = 0,\ \bar{b}_{p0} = \bar{b}_{pN} = 0,\ \bar{a}_{0k} = Y_0(\bar{x}_k),\ \bar{b}_{Nk} = 0, \quad k,p = 0,1,\ldots,N. \end{cases} \tag{2.9}$$

By solving system(2.9), we can obtain approximate optimal solutions. Also, by (2.4) the approximate optimal control and optimal value of objective function can be given as

$$\begin{aligned} U(\bar{t},\bar{x}) &= \frac{-1}{\alpha}\sum_{i=0}^{N}\sum_{j=0}^{N}\bar{b}_{ij}L_i(\bar{t}),L_j(\bar{x}) \\ J(y,u) &= \frac{T}{8}\sum_{k=0}^{N}\sum_{p=0}^{N}w_k w_p[(a_{pk} - z(t_p,x_k))^2 + \alpha c_{pk}^2], \end{aligned} \tag{2.10}$$

where, $c_{pk} = -\frac{1}{\alpha}b_{pk}$ and $w_s, s = 0,1,...,N$ are the quadrature weights of the integral approximation (2.10).

Example 2.1. Consider problems (2.1)-(2.2), where $T = 1$, $\alpha = 1$, $\nu = 0.01,\ 0.05$, $y_0 = sin(4\pi x)$, $\Phi(u) = u$ and $z(t,x) = 0$. The approximate optimal value of objective function computed by the CPS method for $\nu = 0.01,\ 0.05$ and $N = 10,\ 20,\ 30$ and 40 are shown in Table 1. We observe that our numerical results are better than the results of LPS

method [1]. In Figure 2, we show the obtained approximate optimal state and control for $\nu = 0.05$ and $N = 40$.

TABLE 1. Comparison of objective function values for Example 2.1.

	$\nu = 0.01$	$\nu = 0.01$	$\nu = 0.05$	$\nu = 0.05$
N	Presented method	LPS method[1]	Presented method	LPS method[1]
10	0.033014881174862	0.0828638100277	0.016911085761598	0.01590006952876
20	0.040440356851832	0.0620867108909	0.013730446517693	0.01519309308076
30	0.029728232559725	0.0466282421253	0.015082505388501	0.01519227846689
40	0.029005091596013	0.0463124455511	0.015073940981453	0.01519176630695

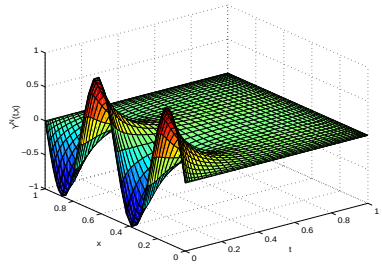

Figure 2(a) The approximate optimal state for$N = 40$ and $\nu = 0.05$

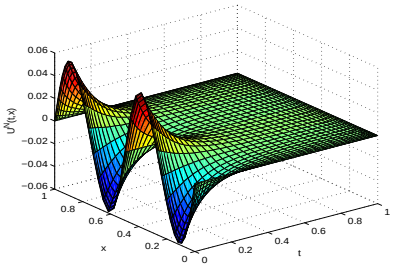

Figure 2(b) The approximate optimal control for$N = 40$ and $\nu = 0.05$

3. Conclusion

The optimal control of Burgers equation is known as a complex problem in control theory. In this work, by applying CPS method for optimal control of Burgers equation, we achieved a good approximate optimal solution with good accuracy.

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AN INTERIOR POINT METHOD FOR SOLVING LINEAR OPTIMIZATION BASED ON A NEW GENERIC KERNEL FUNCTION

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Abstract. In this paper, first an interior point algorithm based on a new class of kernel functions obtained by imposing some mild conditions on the kernel function is given. This class is fairly general and includes the SR functions, non-SR functions, the classical logarithmic functions, as well as trigonometric functions. Then, we compute the worst case iteration complexity bounds for the new generic kernel function. Finally, we define three new kernel functions that are not presented so far the literature and show that the primal-dual IPM based on these functions enjoys the best known complexity bound for large-update methods.

1. Introduction

We consider the standard form of Linear Optimization (LO) problem as:

$$(P) \qquad \min\{c^T x : Ax = b, x \geq 0\},$$

and hence, the dual problem of (P) is given by:

$$(D) \qquad \max\{b^T y : A^T y + s = c, s \geq 0\},$$

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Key words and phrases. Interior point methods, large-update methods, generic kernel function, complexity bounds.

* Speaker.

where $A \in \mathbf{R}^{m\times n}$, $x, c, s \in \mathbf{R}^n$ and $y, b \in \mathbf{R}^m$. Here, we consider the following assumptions.

- $A1$ (Interior Point Condition (IPC)): There exist a strictly feasible points for (P) and (D), namely, $x^0 > 0$ and (y^0, s^0) with $s^0 > 0$, so that: $Ax^0 = b$ and $A^T y^0 + s^0 = c$.
- $A2$: The matrix A has full row rank, that is $rank(A) = m \leq n$.

Due to the Karush-Kuhn-Tucker (KKT) conditions, finding an optimal solution for the problems (P) and (D) is equivalent with solving the following nonlinear system:

$$Ax = b, \quad A^T y + s = c, \quad xs = 0, \qquad (x, s) \geq 0. \tag{1.1}$$

The basic idea of primal-dual Interior Point Methods (IPMs) for LO problems is to replace the equation $sx = 0$ in (1.1), by the parameterized equation $xs = \mu\mathbf{e}$, where μ is a real positive parameter and $\mathbf{e} = (1, 1, \ldots, 1)^T$. Therefore, the new system is given by:

$$Ax = b, \qquad A^T y + s = c, \qquad xs = \mu\mathbf{e}, \qquad (x, s) \geq 0. \tag{1.2}$$

Due to A1 and A2, this new system has a unique solution [1]. A direct application of the Newton method on (1.2) provides the following system for Δx, Δy and Δs:

$$A\Delta x = 0, \qquad A^T \Delta y + \Delta s = 0, \qquad x\Delta s + s\Delta x = \mu\mathbf{e} - xs. \tag{1.3}$$

For simplicity, let us define the scaled vector v and new search directions d_x and d_s as $v := \sqrt{\frac{xs}{\mu}}$, $d_x = \frac{v\Delta x}{x}$ and $d_s = \frac{v\Delta s}{s}$.
Therefore, the system (1.3) is converted to the following one:

$$\bar{A}d_x = 0, \quad \bar{A}^T \Delta y + d_s = 0; \qquad d_x + d_s = v^{-1} - v, \tag{1.4}$$

where $\bar{A} := \frac{1}{\mu} AV^{-1}X$, $V := diag(v), X := diag(x)$ and $S := diag(s)$
One can easily see that $v - v^{-1}$ equals to the minus gradient of the following *proximity function* [4]: $\Psi_c(v) := \sum_{i=1}^n \psi_c(v_i), \quad v_i \in R_{++}$, where $\psi_c(t) = \frac{t^2-1}{2} - \log(t)$, is so-called *kernel function* of the proximity function. We can convert the system (1.4) to the following system:

$$\bar{A}d_x = 0, \quad \bar{A}^T \Delta y + d_s = 0, \quad d_x + d_s = -\nabla\Psi(v), \tag{1.5}$$

where, $\Psi(v) = \sum_{i=1}^n \psi(v_i)$ is a proximity function.
Recently, several interior point methods based on the kernel function have been constructed. An important work in this direction goes back to work proposed by Bai et al. [1]. Moreover, we can find interesting works with interior point methods based on the trigonometric kernel function in [2, 3, 4]

2. Complexity results

Let us define the new class of functions as:

$$\psi(t) = \frac{t^2-1}{2} - \int_1^t \exp(f(x))dx, \tag{2.1}$$

where $f(x)$ is a twice differentiable on $(0,\infty)$ that satisfies the following conditions:

$$f(1) = 0, \ \lim_{t\mapsto 0^+} f(t) = +\infty, \quad tf'(t) + 1 < 2t\exp(-f(t)), \tag{2.2}$$

$$f'(t) < 0 \quad \text{and } f''(t) > 0, \ \forall\ t > 0. \tag{2.3}$$

By using (2.1) and (2.2), we have: $\lim_{t\to+\infty}\psi(t) = \lim_{t\to+0^+}\psi(t) = +\infty$, which means that the function $\psi(t)$ has barrier property [1].

Lemma 2.1. *Let $t > 0$; thus, for the function $\psi(t)$ defined by (2.1), we have:* 1) $\psi'''(t) < 0$, 2) $t\psi''(t) + \psi'(t) > 0$, 3) $\psi''(t) > 1$.

Now, we define the norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) := \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(v_i))^2}, \qquad v \in R^n_{++}. \tag{2.4}$$

In the following, we present a technical lemma, which is useful to compute an upper bound for the proximity function.

Lemma 2.2 (Lemma 2.2 in [3])**.** *For the new kernel function $\psi(t)$ defined by (2.1), we have:*

$$1)\ \Psi(v) \le 2\delta(v)^2, \quad 2)\ \|v\| \le \sqrt{n} + \sqrt{2\Psi(v)} \le \sqrt{n} + 2\delta(v).$$

An upper bound for total number of iterations is given by the following theorem.

Theorem 2.3. *Assume that $\tau = O(n) \ge 1$; therefore, the total number of iterations to get an ϵ solution (i.e., a solution that satisfies $x^T s = n\mu \le \epsilon$) is given by $\left\lceil \frac{n^\gamma}{\kappa\gamma} \right\rceil \left\lceil \frac{1}{\theta}\log\frac{n}{\epsilon} \right\rceil$, where $\kappa > 0$ and $0 < \gamma \le 1$.*

Proof. The proof is similar to the proof of Theorem 6.1 in [4]. Therefore, we omit it here. □

3. Three New kernel functions

In this section we give three new kernel functions which are not presented so far in literatures. To start, we define three function $f_i(t)$'s, which satisfy the conditions (2.2)–(2.3). The functions $f_i(t)$ and new kernel functions are defined in Table 1.

TABLE 1. Three new kernel functions

i	$f_i(x)$	$\psi_i(t)$
1	$p(\frac{e-1}{e^x-1}-1)$	$\frac{t^2-1}{2}-\int_1^t e^{p(\frac{e-1}{e^x-1}-1)}dx$
2	$p\log(\frac{e-1}{e^x-1})+p(\frac{e-1}{e^x-1}-1)$	$\frac{t^2-1}{2}-\int_1^t (\frac{e-1}{e^x-1})^p e^{p(\frac{e-1}{e^x-1}-1)}dx$
3	$-2p\log(x)+p(\tan(h(x))-1)$	$\frac{t^2-1}{2}-\int_1^t \frac{1}{x^{2p}}e^{p(\tan(h(x))-1)}dx$

In Table 1, the function $h(t)$ is defined as $h(t) := \frac{\pi}{2+2t}$. By using information presented in the previous section, we conclude that the new kernel functions have the best known iteration complexity bounds, i.e., $O(\sqrt{n}\log n\log\frac{n}{\epsilon})$. We perform interior point algorithm based with new kernel functions on the a test problem in [2]. The obtained results are presented in Table 2.

TABLE 2. The numerical results of performing interior point Algorithm for $m = 1000$.

	θ	0.05	0.1	0.15	0.2	0.25	0.3	0.4	0.5
ψ_1	Iter	34	31	29	30	30	31	25	26
	Time	154.78	79.98	56.18	45.00	37.01	31.09	27.86	26.23
ψ_2	Iter	35	29	28	28	27	26	24	26
	Time	231.12	79.81	51.79	39.67	32.79	26.74	20.74	17.01
ψ_3	Iter	26	24	29	22	22	20	21	18
	Time	163.75	85.92	43.59	45.17	38.10	32.95	26.75	21.60

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SOLVING OF VOLTERRA OPTIMAL CONTROL PROBLEM USING SECOND CHEBYSHEV WAVELET METHOD

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Abstract. This paper presents an approximate method for solving a class of optimal control problems (OCP) of Volterra integral equations. The method is based on using the second Chebyshev wavelet (SCW) functions. We give the SCW operational matrix, and combine it with the block pulse functions (BPFs) to derive the procedure of solving this kind of OCPs. In this method, we do not need any projection method and any integrations to obtain the coefficient of SCW expansion.

1. Introduction

Many problems in economics, biology, epidemiology, and memory effects can be modelled as a Volterra optimal control problem (VOCP) which are solvable by dynamic programming methods. There are different technique for solving the optimal control problem governed by Volterra integral equations. Of all of them, orthogonal functions have received considerable attention dealing with various OCPs. For example, [1] has been presented a TFs method for solving VOCPs. Or the necessary and sufficient conditions on the existence solution of VOCPs have been considered in [2].

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* Speaker.

In this paper, we propose a new method to solve numerically the following OCP by SCW

$$\begin{aligned} &\text{Min } J = F_0(x(t_f)) + \int_0^{t_f} F(t, x(t), u(t))dt \\ &s.t. \\ &\quad \begin{cases} x(t) = g(t) + \int_0^t K(t, s, x(s), u(s))ds, \\ x(0) = x_0, \end{cases} \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, is the state vector and the vector function $u \in \mathbb{R}^m$ is the control function.

2. SCW and its properties

2.1. **Construction of SCW.** The second Chebyshev wavelets, defined on the interval $[0, 1)$, have the following form [3, 4],

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{U}(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $n = 1, \cdots, 2^{k-1}$, k is any positive integer, and $\tilde{U}_m(t) = \sqrt{\frac{2}{\pi}}U_m(t)$, that $U_m(t)$ is the second Chebyshev polynomial of degree m which respect to the weight function $w(t) = \sqrt{1-t^2}$. They are defined on $[-1, 1]$ by the recurrence:

$$U_0(t) = 1,\ U_1(t) = 2t,\ U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t),\ m = 1, 2, \cdots.$$

The weight function $\tilde{\omega}(t) = \omega(2t - 1)$ has to be dilated and translated as $\omega_n(t) = \omega(2^k t - 2n + 1)$.

2.2. **Function approximation.** A function $f(t)$ defined over $[0, 1)$, may be expressed in terms of the SCW as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t),$$

where the coefficient vector C and SCW function vector Ψ are given by:

$$C = [c_{10}, \cdots, c_{1M-1)}, c_{20}, \cdots, c_{2M-1)}, \cdots, c_{2^{k-1}0}, \cdots, c_{2^{k-1}M-1)}]^T,$$

$$\Psi(t) = [\psi_{10}, \cdots, \psi_{1M-1)}, \psi_{20}, \cdots, \psi_{2M-1)}, \cdots, \psi_{2^{k-1}0}, \cdots, \psi_{2^{k-1}M-1)}]^T.$$

Taking the collocation points as $t_i = \frac{2i-1}{2^k M}$, $i = 1, 2, \cdots, 2^{k-1}M$, we define the SCW matrix

$$\Phi_{m' \times m'} = [\Psi(\frac{1}{2m'}), \Psi(\frac{3}{2m'}), \cdots, \Psi(\frac{2m'-1}{2m'})],$$

where $m' = 2^{k-1}M$. Next, we define a m-set of block pulse functions (BPFs) as

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m} \leq t < \frac{(i)}{m}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

where $i = 1, 2, \cdots, m$. The SCW may be expanded into an m-term BPFs as

$$\Psi_m(t) = \Phi_{m\times m} B_m(t), \tag{2.3}$$

where $B_m(t) := [b_1(t), b_2(t), \cdots, b_m(t)]^T$.

Lemma 2.1. *Let $\Psi_{m'}(t)$ be a m'-vector function of the SCW then*

$$\int_0^1 \Psi^T(t)\Psi(t)dt = \frac{1}{m'} trac(\Phi^T_{m'\times m'}\Phi_{m'\times m'}).$$

where for any $m' \times m'$ matrix A, $trac(A) := \sum_{i=1}^{m} a_{ii}$.

Proof: From Eq. (2.3), we can write

$$\begin{aligned} \int_0^1 \Psi^T_{m'}(t)\Psi_{m'}(t)dt &= \int_0^1 (\Phi_{m'\times m'} B_{m'}(t))^T (\Phi_{m'\times m'} B_{m'}(t))dt \\ &= \int_0^1 B^T_{m'}(t)\Phi^T_{m'\times m'}\Phi_{m'\times m'} B_{m'}(t)dt. \end{aligned}$$

Put $A := \Phi^T_{m'\times m'}\Phi_{m'\times m'}$. If a_i shows every column of the matrix A, then

$$\begin{aligned} \int_0^1 \Psi^T_{m'}(t)\Psi_{m'}(t)dt &= \int_0^1 B^T_{m'}(t)[a_1, a_2, \cdots, a_{m'}]B_{m'}(t)dt \\ &= \int_0^1 [B^T_{m'}(t)a_1, B^T_{m'}(t)a_2, \cdots, B^T_{m'}(t)a_{m'}]B_{m'}(t)dt \\ &= \int_0^1 (\sum_{i=1}^{m} b_i(t)B^T_{m'}(t)a_i)dt \\ &= \int_0^1 (\sum_{i=1}^{m} [0, 0, \cdots, b_i(t), 0, \cdots, 0]a_i)dt \\ &= \int_0^1 (\sum_{i=1}^{m} a_{ii}b_i(t))dt \\ &= \frac{1}{m'}\sum_{i=1}^{m} a_{ii}. \end{aligned}$$

3. Approximation method

By expanding the function $x(t)$ and $u(t)$ in terms of the SCW in Eq. (1.1), the Volterra integral equation constraint is converted into the following equation

$$\Theta(X,U) = X^T\Psi(t) - X_0^T\Psi(t) - \int_0^t K(t,s,X^T\Psi(s),U^T\psi(s))ds = 0.$$

We also have the objective function as:

$$J(X,U) = F_0(X^T\Psi(t_f)) + \int_0^{t_f} F(t,X^T\Psi(t),U^T\Psi(t))ds.$$

The optimal control problem is to find X and U such that $J(X,U)$ is minimized subject to the constraint $\Theta(X,U) = 0$. We construct the following Lagrange function as follows:

$$J^*(X,U) = J(X,U) + \lambda\Theta(X,U),$$

where λ is a $2^{k-1}M$-vector and denotes the Lagrange multiplier. The necessary conditions for minimizing J^* are given by:

$$\begin{cases} \frac{\partial J^*}{\partial X} = 0, \\ \frac{\partial J^*}{\partial U} = 0, \\ \frac{\partial J^*}{\partial \lambda} = 0. \end{cases} \tag{3.1}$$

After solving the above nonlinear system in terms of the unknown coefficients of the vectors X, U and λ, the state function $x_{m'}(t)$ and the optimal control $u_{m'}(t)$ will be obtained.

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AN ITERATIVE NEWTON PSEUDOSPECTRAL METHOD FOR SOLVING A CLASS OF OPTIMAL CONTROL PROBLEMS

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Abstract. The objective of this paper is to present a novel method to design a suboptimal controller for a wide class of nonlinear optimal control problems. The proposed method is a combination of a Legendre pseudospectral successive approximation method (PSAM) and Newton method, for solving the extreme conditions derived by Ponteryagin's maximum principle (PMP). An illustrative numerical example is included to demonstrate the accuracy, efficiency and the reliability of the proposed method.

1. Introduction

In the control theory, a major importance is conferred to optimal control problems. This interest is justified by the great number of practical applications in physics, economy, aerospace, chemical engineering, robotic, etc. For the general optimal control problem (OCP), however, an analytical solution does not exist. This has inspired researchers to propose approaches to obtain an approximate solution for it. It is well-known that the OCP leads to a TPBVP obtained from the PMP. Many recent approaches have been devoted to solve this problem. Recently, a growing interest has been appeared toward the application

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of approximate analytical techniques in solving this TPBVP. In [1],[2], the authors give an analytical approximate solution for linear and nonlinear quadratic OCP's using the homotopy perturbation and analysis methods (HPM and HAM). Also, in [3], the basic and a modified VIM are successfully applied to the TPBVP, obtained from nonlinear quadratic OCP's. In this paper, a novel Newton SAM is proposed. We first derive the TPBVP from the PMP and then apply a novel Newton SAM to solve it. This method is applicable for a large class of linear and nonlinear OCP's. The simplicity and the efficiency of the proposed Newton SAM is demonstrated through an illustrative example.

2. Statement of the OCP and optimality conditions

Consider the following optimal control problem:

$$\begin{array}{l} J[x,u] = \frac{1}{2}\int_0^{t_F}(Q(x(t)) + u^T(t)Ru(t))dt \\ \dot{x}(t) = f(t,x(t)) + g(t,x(t))u(t), \quad t \in [0,t_F] \\ x(0) = x^0. \end{array} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are denoting the state variable and control variables, and x^0 the given initial state at $t = 0$. $Q(x(t))$ a positive semi-definite real function and $R \in \mathbb{R}^{m\times m}$ a positive definite matrix. Our aim is to minimize the objective functional The known extreme necessary conditions are:

$$\begin{array}{l} \dot{x} = f(t,x) + g(t,x)[-R^{-1}g^T(t,x)\lambda] \\ \dot{\lambda} = -\left(\frac{1}{2}\nabla Q(x) + (\frac{\partial f(t,x)}{\partial x})^T\lambda + \sum_{i=1}^n \lambda_i[-R^{-1}g^T(t,x)\lambda]^T\frac{\partial g_i(t,x)}{\partial x}\right) \\ x(0) = x^0, \lambda(t_F) = 0. \end{array} \tag{2.2}$$

where $\lambda(t) \in \mathbb{R}^m$ is the co-state vector and the optimal control law is obtained by $u^* = -R^{-1}g^T(t,x)\lambda$. For convenience, let $X(t) = [X_1, ..., X_{n+m}] := [x(t); \lambda(t)]$ and define the right hand sides of (2.2) as,

$$\Psi(t,x,\lambda) := \begin{bmatrix} f(t,x) + g(t,x)[-R^{-1}g^T(t,x)\lambda] \\ -\left(\frac{1}{2}\nabla Q(x) + (\frac{\partial f(t,x)}{\partial x})^T\lambda + \sum_{i=1}^n \lambda_i[-R^{-1}g^T(t,x)\lambda]^T\frac{\partial g_i(t,x)}{\partial x}\right) \end{bmatrix} \tag{2.3}$$

Thus the TPBVP in (2.2) can be rewritten in the operator form as:

$$\begin{array}{l} \mathcal{F}_r[X(t)] := \mathcal{L}_r[X(t)] + \mathcal{N}_r[X(t)] = 0, \quad r = 1,2,...,n+m, \\ X_{1:n}(0) = x^0, \quad X_{n+1:n+m}(t_F) = 0, \end{array} \tag{2.4}$$

where $\mathcal{L}_r$ and $\mathcal{N}_r$ are linear and nonlinear operators.

3. A Novel Pseudospectral Newton SAM

In this section, we propose a novel SAM to solve the TPBVP in (2.4). Construct a sequence of solutions for solving (2.4), as follows:

$$\mathcal{L}_r[X_{k+1}(t)] = -\mathcal{N}_r[X_k(t)], \quad r = 1, 2, ..., n+m, \tag{3.1}$$

$$\mathcal{F}'_r[X_{k+1}(t)]\,(X_{k+2}(t) - X_{k+1}(t)) = \mathcal{F}_r[X_{k+1}(t)], \tag{3.2}$$

for which $k \geq 0$, the first n entries of $X_{k+1}(t)$ at $t = 0$ are x^0 and its last n entries at $t = t_F$ are 0.

Let $L_i(t)$ be the shifted Legendre polynomials for $t \in [0, t_F]$ and t_j^N, $0 \leq j \leq N$, be the Legendre-Gauss-Lobatto (LGL) points. Suppose $X_{r,k}(t) \in \mathbb{R}$ be the rth component of the unknown vector function $X_k(t)$. Then $X_{r,k}(t)$ can be approximated by means of the Legendre basis polynomials up to order N, $X_{r,k}(t) \approx X_{r,k}^N(t) = \sum_{j=0}^{N} L_j(t) X_{r,k}^{N,j}$, where $X_{r,k}^{N,j}$ is the unknown coefficient of the Legendre polynomial of degree j, $L_j(t)$. To approximate the derivatives of the unknown function $X_{r,k}(t)$ at the collocation points, we use the Legendre spectral differentiation matrix $\mathbf{D}$ as the matrix vector product

$$\dot{X}_{r,k}(\mathbf{t}^N) \approx \dot{X}_{r,k}^N(\mathbf{t}^N) = \mathbf{D}\mathbf{Y}_{r,k}^N, \tag{3.3}$$

where $\mathbf{t}^N = [t_0^N, t_1^N, ..., t_N^N]^T$, and $\mathbf{Y}_{r,k}^N = [X_{r,k}^{N,0}, X_{r,k}^{N,1}, \ldots, X_{r,k}^{N,N}]^T$ is the vector of function $X_{r,k}(t)$ values at the collocation points and $\mathbf{D} = 2D/t_F$ where D is an $(N+1) \times (N+1)$ matrix whose entries are defined as in [4]. Now, we substitute the approximate solution, $X_{r,k}^N(t)$, into (3.1)-(3.2) and require that it satisfy the equations at the LGL nodes. This requirement generates the following pseudospectral SAM (PSAM):

$$\mathbf{A}\mathbf{W}_{k+1}^N = -\mathbf{N}[\mathbf{W}_k^N], \quad k \geq 0, \tag{3.4}$$

$$\mathbf{F}'^N_{k+1}(\mathbf{W}_{k+2}^N - \mathbf{W}_{k+1}^N) = -\mathbf{F}_{k+1}^N, \tag{3.5}$$

$$\mathbf{W}_{k+1,1:n}^N(t_0^N) = x^0, \quad \mathbf{W}_{k+1,n+1:n+m}^N(t_N^N) = 0, \tag{3.6}$$

where $\mathbf{N}[\mathbf{W}_k^N]$ is an $(N+1)(n+m)$ column vector whose $\mathbf{N}_r[\mathbf{W}_k^N]$ corresponds to $\mathcal{N}_r[X_{r,k}(t)]$ when evaluated at the collocation points for any $r = 1, 2, \ldots, N$. The matrix $\mathbf{A}$ is an $((N+1)(n+m))^2$ square block matrix which is derived from transforming the linear operators $\mathcal{L}_r$,$r = 1, \ldots, n+m$, at LGL collocation nodes, using the derivative matrix $\mathbf{D}$ and defined as $\mathbf{A} = (A_{r,i})$,

$$A_{r,i} = \begin{cases} \mathbf{D} + p_{r,i}(\mathbf{t}^N)^T\mathbf{I}, & r = i, \\ p_{r,i}(\mathbf{t}^N)^T\mathbf{I}, & r \neq i, \end{cases}$$

TABLE 1. The maximum error of PSAM for $x_1(t)$ with $N = 20$, compared to VIM [3] and HAM [2].

k	CPU time (sec.)	Max error NSAM	CPU time (sec.)	Max error VIM	CPU time (sec.)	Max error HAM
2	0.018822	4.8041e-04	0.047	5.1463e-1	0.34972	6.5807e-2
3	0.023131	6.2656e-05	0.094	1.7670e-1	1.19685	6.0841e-2
4	0.040204	6.2633e-05	0.109	1.3528e-1	3.05910	5.2627e-2

where $\mathbf{I}$ is an identity matrix of order $N + 1$. Also, $\mathbf{F}$ and $\mathbf{F}'$ can be defined in a similar manner.

4. Illustrative example

Consider the two-dimensional nonlinear composite system described by

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1^3 + x_2^2 + u_1 \\ \dot{x}_2 &= -x_2 + x_2(x_1 + x_2^2) + u_2 \\ x_1(0) &= 0, \quad x_2(0) = 0.8. \end{aligned}$$

The quadratic cost functional to be minimized is given by:

$$J = \frac{1}{2}\int_0^1 (x_1^2 + x_2^2 + u_1^2 + u_2^2)dt.$$

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ON SLIDING MODE CONTROL VIA STOCHASTIC PERTURBATION

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Abstract. This paper is concerned with the design of sliding mode control (SMC) for a class of stochastic systems with Markovian jump systems (MJSs). It is assumed that the transmitted information may be lost, and the probability distribution of packet dropout obeys Bernoulli process. It is shown that SMC function can be driven on to the specified sliding surfaces for each mode in finite time and system is stable. In addition, we simulate and solve the problem with MATLAB. Finally, a numerical example is given.

1. Introduction

Sliding mode control (SMC), as an effective robust control strategy, has been successfully applied to a wide variety of engineering, including uncertain systems, stochastic systems and Markovian jump systems (MJSs) . This paper will be concerned with the design of SMC for a class of stochastic systems with MJSs. It is assumed that the transmitted information may be lost, and the probability distribution of packet dropout obeys Bernoulli process [2]. In this paper Firstly, a sliding-mode surface is constructed, Secondly an estimation method is

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* Speaker.

proposed to cope with the packet losses, based on sliding surface is chosen, and a dropout-probability dependent SMC law is designed. Finally we simulate and solve it problem with MATLAB and a numerical example is given.

Notation : In this paper, $\|.\|$ denotes the Euclidean norm. For a real matrix, $M > 0$ means that M is symmetric and positive definite, and I is identity matrix. The symbol $'*'$ denotes a term that is induced by symmetry. (Ω, F, P) is a probability space with Ω the sample space, and F the σ-algebra of subsets of the sample space, and P is the probability measure. $\varepsilon\{.\}$ denotes the expectation operator.

2. Main results

2.1. **Problem definition.** Consider the discrete-time MJSs with stochastic perturbation:

$$x(k+1) = A(r_k)x(k) + B(r_k)u(k, r_k) + f(x(k), k) + (C(r_k) + \Delta C(r_k))x(k)w(k) \quad (2.1)$$

Where $x(k) \in \mathbb{R}^n$ is the system state, $u(k, r_k) \in \mathbb{R}^m$ is the control input,$w(k)$ is a scalar Wiener process on a probability space (Ω, F, P) relative to an increasing family $(F_k)_{k\in\mathbb{N}}$ of σ-algebra $F_k \subset F$ generated by $(w(k))_{k\in\mathbb{N}}$ with $\mathbb{N}$ the set of natural numbers, and is The unknown nonlinear function $f(x(k), k)$, is the external disturbance with known constant bound. Let $\{r_k, k \geq 0\}$be a process on the finite state space $l = \{1, 2, ..., N\}$, and governs the switching among the different system modes, whose mode transition probabilities are given as:

$$p\{r_{k+1} = j \mid r_k = i\} = \pi_{ij} \quad (2.2)$$

Where the $\pi_{ij} > 0$ with $i \neq j$,is transition rate from mode i to mode j. and $\sum_{j=1}^{N} \pi_{ij} = 1$, and the transition probability matrix is defined as $\Pi = (\pi_{ij})_{i,j=1:N}$. for each $r_k = i \in l$, we compose the matrix $A(r_k) = A_i$,$B(r_k) = B_i$, $C(r_k) = C_i$ and $\Delta C(r_k) = \Delta C_i$ for i-th mode. The unknown matrix ΔC_i $\Delta C_i = E_i F_i(k) H_i$, where E_i and H_i are known real constant matrices, and $F_i(k)$ is an unknown matrix satisfying $F_i(k)^T F_i(k) \leq I$ for any $k \in l$. Then The system (1) becomse

$$x(k+1) = A_i x(k) + B_i(u(k, i) + f(x, k) + (C_i + \Delta C_i)x(k)w(k) \quad (2.3)$$

It is assumed that the system states may be lost when transmitted from sensor to the controller, and the probability distribution of the packet dropout obeys Bernoulli process $\theta \in \mathbb{R}$ as follows:

$$P\{\theta = 1\} = \bar{\theta}, P\{\theta = 0\} = 1 - \bar{\theta}, 0 \leq \bar{\theta} \leq 1 \quad (2.4)$$

$\bar{\theta}$ is probability that any data packet will be lost. It is assumed that the packet dropouts may happen successively. In order to compensate the lost packet, the following method will be utilized as:

$$x_s(k) = (1-\theta)x(k) + \theta x_s(k-1) \tag{2.5}$$

Which is termed as compensator. Now, the objective of this work is to design a SMC law for the MJSs with stochastic perturbation.

2.2. **Design of sliding controller.** We will design an sliding-mode controller as:

$$s(k,i) = (1-\bar{\theta})G_i x(k) + \bar{\theta} G_i A_i x_s(k-2) \tag{2.6}$$

Where $\bar{\theta}$ is the dropout probability in(2.4) and $G_i = B_i^T \bar{P}_i$ that B_i define in(2.3) and $\bar{P}_i = \sum_{j=1}^{N} \pi_{ij} P_j$ for all $i \in l$ .The matrices $P_j > o$,$j \in l$ will be determined in Theorem 2 . Suppose $\phi = \bar{\theta}/(1-\bar{\theta})$, which substituted into (2.3) may yield the sliding mode dynamics in the sliding surface $s(i,k) = 0$ as follows:

$$\begin{aligned} x(k+1) = [A_i - B_i(G_iB_i)^{-1}G_iA_i]x(k) - \phi B_i(G_iB_i)^{-1}G_iA_ix_s(k-1) \\ +[I - B_i(G_iB_i)^{-1}G_i](C_i + \Delta C_i)x(k)w(k) \end{aligned} \tag{2.7}$$

2.3. **Sliding mode control and reachability.** Define $D(k) = (1-\bar{\theta})G_iB_if(x(k),k) = [d_1(k)\ d_2(k)\dots d_m(k)]^T$. The nonlinear function vector$f(x(k),k)$ is bounded, there exist known constants $\underline{d}_i \le d_i(k) \le \bar{d}_i$, Then, let $D_o = [d_{1o}\ d_{2o}\ \dots d_{mo}]^T$,$D_s = [d_{1s}\ d_{2s}\ \dots d_{ms}]^T$ that:

$$d_{io} = \frac{\bar{d}_i + \underline{d}_i}{2},\ d_{is} = \frac{\bar{d}_i - \underline{d}_i}{2}\ (i = 1,2,\dots,m), \tag{2.8}$$

By means of the information from compensator (2.5) and the bounds in (2.8), the desired SMC law is designed as follows:

$$u(k) = -\frac{1}{1-\bar{\theta}}(G_iB_i)^{-1}[G_iA_ix_s(k) + D_o + D_s sgn(S_s(k,i)), \tag{2.9}$$

Where $s_s(k,i)$ is $s(k,i)$ in (2.6) with $x(k)$ replaced by $x_s(k)$.

Theorem 2.1. [1] *For the system* (2.3) *and subject to packet dropout* (2.4)*, if there exist symmetric matrices* $P_i > 0$,$Q_i > 0$, *and scalar*

$\epsilon_i > 0, 0 < \gamma_i < 1$ *satisfying the following LMI:*

$$\begin{bmatrix} -P_i + \epsilon_i H_i^T H_i & * & * & * & * & * & * & * & * \\ 0 & -Q_i & * & * & * & * & * & * & * \\ K_{1i} & K_{2i} & K_{3i} & * & * & * & * & * & * \\ \bar{P}_i C_i & 0 & 0 & -\bar{P}_i & * & * & * & * & * \\ K_{4i} & K_{5i} & 0 & 0 & K_{6i} & * & * & * & * \\ (1-\bar{\theta}) G_i C_i & 0 & 0 & 0 & 0 & -I & * & * & * \\ (1-\bar{\theta}) \bar{Q}_i & \theta \bar{Q}_i & 0 & 0 & 0 & 0 & -\bar{Q}_i & * & * \\ \bar{Q}_i & -\bar{Q}_i & 0 & 0 & 0 & 0 & 0 & -\alpha^{-2}\bar{Q}_i & * \\ 0 & 0 & 0 & E_i^T \bar{P}_i & 0 & E_i^T \bar{P}_i B_i & 0 & 0 & -\epsilon_i I \end{bmatrix} < 0 \tag{2.10}$$

With $\phi = \bar{\theta}/(1-\bar{\theta})$, $\bar{\phi} = (4+\phi)^{1/2}$, $\hat{\phi} = (2\phi + 2\phi^2)^{1/2}$, $K_{1i} = [0 \bar{\phi} A_i^T \bar{P}_i \bar{\phi} A_i^T G_i^T 0]^T$, $K_{2i} = [0\ 0\ 0\ \hat{\phi} A_i^T G_i^T]^T$, $K_{3i} = diag\{\gamma_i I, \bar{P}_i, G_i B_i, G_i B_i\}$, $k_{4i} = [2\alpha A_i^T\ G_i^T\ 0]$, $k_{5i} = [0\ 2\alpha A_i^T\ G_i^T]$, $K_{6i} = diag\{2I, 2I\}$, *Then the SMC law* (2.8) *can ensure that the state trajectories are driven into a band of the sliding surface specified by* (2.6) *with* $G_i = B_i^T P_i$.

2.4. **Simulation example.** Consider the stochastic system (2.1) with two modes and parameters as follows:

$$A_1 = \begin{bmatrix} 0.05 & 0.1 & -0.2 \\ -0.1 & 0.1 & -0.1 \\ 0 & 0.1 & -0.2 \end{bmatrix}, B1 = \begin{bmatrix} -1.2 & -2.5 \\ 1.5 & 3.0 \\ 5.8 & 3.5 \end{bmatrix} C_1 = \begin{bmatrix} -.3 & 0.1 & 0.3 \\ 0.2 & 0.1 & -0.1 \\ 0.3 & -0.4 & -0.2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.1 & -0.3 & -0.2 \\ -0.2 & 0.1 & -0.2 \\ 0.1 & -0.2 & -0.2 \end{bmatrix}, B2 = \begin{bmatrix} 1.0 & 2.0 \\ -1.1 & 1.5 \\ 3.4 & 2.7 \end{bmatrix} C_2 = \begin{bmatrix} -0.2 & 0.1 & 0.3 \\ 0.2 & -0.1 & -0.1 \\ 0.2 & 0.3 & -0.4 \end{bmatrix}$$

$\pi_{11} = 0.3, \pi_{12} = 0.7, \pi_{21} = 0.6, \pi_{22} = 0.4$,$E_1 = E_2 = H_1 = 0.5 H_2 = [0.10.10.1]^T, \bar{\theta} = 0.2$ To solve the LMIs in (2.10) via Matlab LMI toolbox, we can obtain the P_1, P_2, Q_1, Q_2 matrixs and $G_i = B_i^T \bar{P}_i$ in sliding function (2.6). The simulation results show that after 20 second time the proposed sliding mode controller can effectively cope with the effect of Markovian switching and packet losses, and ensure the exponentially mean-square stable of the overall closed-loop system successfuly.

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A GRADIENT DESCENT METHOD FOR SOLVING QUADRATIC OPTIMAL CONTROL OF NONLINEAR SYSTEMS

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ABSTRACT. In this paper, we obtain the gradient of objective function using the concept of directionally derivative. Then the gradient decent method is presented to solve the nonlinear optimal control problem with free final state.

1. INTRODUCTION

Consider the controlled nonlinear dynamical system of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + B(t)u(t), \\ x(t_0) &= x_0, \end{aligned} \tag{1.1}$$

where $x(.) \in \mathbb{R}^n$ denotes the state variable, $u(.) \in \mathbb{R}^m$ denotes the control variable for $t \in [t_0, t_f]$. The function $f(x) \in \mathbb{R}^n$ is a continuously differentiable in all arguments and $B(t) \in \mathbb{R}^{m\times m}$ is given with continuous elements. It is desired to find the control $u(t)$ that minimizes the objective quadratic functional

$$J(x,u) := \frac{1}{2}x^T(t_f)Ax(t_f) + \frac{1}{2}\int_{t_0}^{t_f} x^T(t)Qx(t) + u^T(t)Ru(t)dt, \tag{1.2}$$

subject to dynamical system (1.1) and $u \in U_{ad}$. Here $Q, A \in \mathbb{R}^{n\times n}$ are given positive semi-definite matrices and $R \in \mathbb{R}^{m\times m}$ is a positive definite matrix.

The paper is organized as follows: In the next section, we state the gradient of objective functional. In Section 3, a gradient decent method

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* Speaker.

is introduced for solving first order necessary optimality conditions for $\bar{u}$.

2. The gradient of objective functional

For the derivation gradient of objective functional we introduce the concept of differentiability in functional spaces. Let $(U, \|.\|_U)$ and $(V, \|.\|_V)$ will be tow normed spaces.

All definition and theorems of this section is given from [2].

Definition 2.1. Let $F : U \to V$ be a mapping, $u \in U$ and $h \in U$. If the limit

$$\delta F(u, h) := \lim_{s \to 0} \frac{F(u + sh) - F(u)}{s},$$

exists in V, then it is called the directional derivative of F at u in the direction h.

Definition 2.2. If $\delta F(u, h)$ exists for all $h \in U$ and $\delta F(u, h)$ is a continuous linear operator from U to V, then it is written in the form $\delta F(u)h$, an F is said to be Gateaux differentiable at u. Hence $\delta F(u)$ is shown with $F'(u)$.

Theorem 2.3. *(Chain rule) Let $F : U \to V$ and $G : V \to Z$ be a Gateaux differentiable at u and $F(u)$, respectively. Then $E(u) = G(F(u))$ is also Gateaux differentiable at u and $E'(u) = G'(F(u))F'(u)$.*

Let C be a non empty subset of U and $f : C \subset U \to \mathrm{R}$ a given functional bounded from below. Consider the following optimization problem:

$$\min_{u \in C} f(u). \tag{2.1}$$

Theorem 2.4. *Suppose that $\bar{u} \in C$ is a local minimum of (2.1) and $v - \bar{u}$ is an admissible direction. If $f(.)$ is directionally differentiable at $\bar{u}$, in direction $v - \bar{u}$, then $\delta f(\bar{u})(v - \bar{u}) \geq 0$.*

Corollary 2.5. *Let $C = U$ and $\bar{u}$ be a local optimal solution for (2.1), if $f(.)$ is Gateaux differentiable at $\bar{u}$, then*

$$f'(\bar{u})h = \nabla f(\bar{u})h = 0, \ \forall h \in U. \tag{2.2}$$

If $f(.)$ is a Lipschitz function then for any given $u(.)$ there exists a unique solution $x(u)$ to dynamical system (1.1).

Definition 2.6. Consider the operator $G : U \to Y, \quad u \to x(u) = G(u)$, which dedicate to each $u \in U$ the solution $x(u)$ to dynamical system (1.1). The operator $G(u)$ is called as solution operator or control to state operator.

Using this operator, the nonlinear quadratic optimal control problem (1.1) - (1.2) reduces to the following quadratic optimization problem

$$\min_{u\in U_{ad}} f(u) := \min \frac{1}{2}x^T(u)(t_f)Ax(u)(t_f) + \frac{1}{2}\int_{t_0}^{t_f} x^T(u)(t)Qx(u)(t) + u^T(t)Ru(t)dt. \tag{2.3}$$

Suppose $\bar{u}$ is a local optimal solution for (2.3), from corollary (2.5), the following necessary condition will be acquired:

$$f'(\bar{u})h = J_x(x(\bar{u}),\bar{u})x'(\bar{u})h + J_u(x(\bar{u}),\bar{u})h = 0, \tag{2.4}$$

where

$$J_x(x(\bar{u}),\bar{u})x'(\bar{u})h = x^T(\bar{u})(t_f)Ax'(\bar{u})h(t_f) + \int_{t_0}^{t_f} x^(\bar{u})Qx'(\bar{u})hdt, \tag{2.5}$$

$$J_u(x(\bar{u}),\bar{u})h = \int_{t_0}^{t_f} \bar{u}^T Rhdt. \tag{2.6}$$

Using (2.5) and (2.6) in (2.4) we have

$$f'(\bar{u})h = x^T(\bar{u})(t_f)Ax'(\bar{u})h(t_f) + \int_{t_0}^{t_f} x^T(\bar{u})Qx'(\bar{u})hdt + \int_{t_0}^{t_f} \bar{u}^T Rhdt.$$

Definition 2.7. An element $P(.) \in Y$ is called the costate related to $\bar{u}$ if it solves the following equation:

$$\dot{P}(t) = -Qx(\bar{u})(t) - f_x^T(x(\bar{u}))P(t),\ P(t_f) = Ax(\bar{u})(t_f).$$

Now we can proof the following important theorem.

Theorem 2.8. *Let $\bar{u}$ be a loced optimal solution to (2.3) and $\bar{x} = x(\bar{u})$ its corresponding state, then there exists an costate $P(t)$ such that the following systems of equetions is satisfied:*

$$\dot{\bar{x}}(t) = f(\bar{x}(t)) + B\bar{u}(t),\ \bar{x}(t_0) = 0, \tag{2.7a}$$

$$\dot{P}(t) = -Q\bar{x}(t) - f_x^T(\bar{x}(t))P(t),\ P(t_f) = A\bar{x}(t_f) \tag{2.7b}$$

$$\bar{u}(t) = -R^{-1}B^T p(t), \tag{2.7c}$$

System (2.7) is called the first order necessary optimality conditions for $\bar{u}$. Also

$$\nabla_u f(\bar{u}) = p^T B + \bar{u}^T R. \tag{2.8}$$

3. Solving the system (2.7) by gradient decent method

Here we use the gradient decent method to solve optimality system (2.7). Consider the Algorithm 1 that solves the system (2.7) [1].

Algorithm 1 Solving the system (2.7)

1. Choose $u_0 \in U$ and solve the state equation (2.7a) and costate equation (2.7b) to obtain y_0 and p_0.
2. Set k=0
3. repeat
4. Choose the descent direction $d_k = -\nabla f(u_k)$ according to (2.8).
5. Determine $\alpha_k = arg\min_{\alpha>0}\{f(u_k + \alpha d_k)\}$.
6. Set $u_{k+1} = u_k + \alpha_k d_k$ and solve sequentially to obtain (x_{k+1}, p_{k+1})

$$\dot{\bar{x}}_{k+1}(t) = f(\bar{x}_{k+1}(t)) + B\bar{u}_{k+1}(t),\ \bar{x}_{k+1}(t_0) = 0,$$

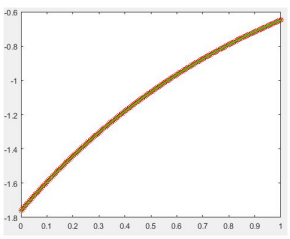
$$\dot{P}_{k+1}(t) = -Q\bar{x}_{k+1}(t) - f_x^T(\bar{x}_{k+1}(t))P_{k+1}(t),\ P_{k+1}(t_f) = A\bar{x}_{k+1}(t_f)$$

7. Set k = k+1.
8. until stopping criteria.

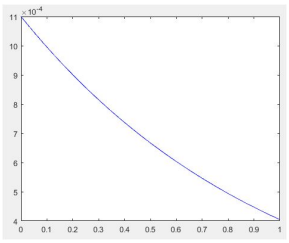
Example 3.1. Consider a the nonlinear problem as following

$$\min\{J = x^2(t_f) + \int_0^{t_f} u^2(t)dt | \dot{x} = x + u,\ x(0) = x_0\} \tag{3.1}$$

Figure (A) shows the approximated and exact solution of control and (B) shows the error of it.



(A)



(B)

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w-EQUITABLE EFFICIENCY IN MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. This paper focuses on solving multiobjective optimization problems by introducing the concept of w-equitable efficiency, where w is a finite decreasing sequence of positive numbers. Moreover, some theoretical and practical aspects of w-equitably efficient solutions are discussed.

1. Introduction

Multiobjective programming has been studied for many years and multiobjective methods have found applications in diverse areas of human life. It is well-known that any multiobjective optimization problem starts usually with an assumption that the criteria are incomparable, i.e., different criteria may have different units and physical interpretations. Many applications, however, arise from situations which present equitable criteria. Equitability is based on the assumption that the criteria are not only comparable (measured on a common scale) but also

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anonymous (impartial). The latter makes the distribution of outcomes among the criteria more important than the assignment of outcomes to specific criteria, and therefore models equitable allocation of resources.

The equitable preference was first known as the generalized Lorenz dominance [3, 5]. Kostreva and Ogryczak [1] are the first ones who introduced the concept of equitability into multiobjective programming. They have shown equitable efficiency to be a refinement of Pareto efficiency by adding, to the reflexivity, strict monotonicity and transitivity of the Pareto preference order, the requirements of impartiality and satisfaction of the principle of transfers. Then Kostreva et al. [2] presented the theory of equitable efficiency in greater generality. More recently, these results are extended by the authors [4].

2. Main results

Throughout this article the following notation is used. Let R^m be the Euclidean vector space and $y', y'' \in R^m$. $y' \leqq y''$ denotes $y'_i \leq y''_i$ for all $i = 1, \ldots, m$. $y' < y''$ denotes $y'_i < y''_i$ for all $i = 1, \ldots, m$. $y' \leq y''$ denotes $y' \leqq y''$ but $y' \neq y''$. The set $\{y \in R^m : y \geqq 0\}$ is denoted by R^m_+.

Consider a decision problem defined as an optimization problem with m objective functions. For simplification we assume, without loss of generality, that the objective functions are to be minimized. The problem can be formulated as follows:

$$\begin{aligned} &\min \{(f_1(x), f_2(x), \ldots, f_m(x))\}, \\ &\textit{subject to } x \in X \end{aligned} \tag{2.1}$$

where x denotes a vector of decision variables selected from the feasible set X and $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is a vector function that maps the feasible set X into the objective (criterion) space R^m_+. We refer to the elements of the objective space as outcome vectors. An outcome vector y is attainable if it expresses outcomes of a feasible solution, i.e., $y = f(x)$ for some $x \in X$. The set of all attainable outcome vectors will be denoted by $Y = f(X)$.

Definition 2.1. Preference relations satisfying the following axioms are called equitable rational preference relations:

1. Reflexivity: for all $y \in R^m$: $y \preceq y$.
2. Transitivity: for all $y', y'', y''' \in R^m$: $y' \preceq y''$ and $y'' \preceq y''' \Rightarrow y' \preceq y'''$.
3. Strict monotonicity: for all $y \in R^m$: $y - \epsilon e_i \prec y$ for $\epsilon > 0$ where e_i denotes the i^{th} unit vector in R^m.

4. Impartiality: for all $y \in R^m$: $(y_1, \ldots, y_m) \simeq (y_{\tau(1)}, \ldots, y_{\tau(m)})$ for any permutation τ.

5. Principle of transfers: for all $y \in R^m$: $y_i > y_j \Rightarrow y - \epsilon e_i + \epsilon e_j \prec y$ for $0 < \epsilon < y_i - y_j$.

We say that outcome vector y' equitably dominates y'' ($y' \prec_e y''$), iff $y' \prec y''$ for all equitable rational preference relations $\preceq$. We say that a feasible solution $x \in X$ is an equitably efficient solution of the multiple criteria problem (2.1), if and only if there does not exist any $x' \in X$ such that $f(x') \prec_e f(x)$. The relation of equitable dominance $\preceq_e$ can be expressed as a vector inequality on the cumulative ordered outcomes.

Definition 2.2. Let $y \in R^m_+$.

1. Let $\Theta : R^m_+ \to R^m_+$ be the ordering map defined as $\Theta(y) = (\theta_1(y), \theta_2(y), \ldots, \theta_m(y))$, where $\theta_1(y) \geq \theta_2(y) \geq \ldots \geq \theta_m(y)$, $\theta_i(y) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$, and τ is a permutation of the set $\{1, 2, \ldots, m\}$.

2. Let $w = (w_j)_{j=1}^m$ be a finite sequence. The ordering map $\Theta_w : R^m_+ \to R^m_+$ is defined by, $\Theta_w(y) = (w_1\theta_1(y), w_2\theta_2(y), \ldots, w_m\theta_m(y))$.

Definition 2.3. We say that outcome vector $y' \in Y$ w-equitably dominates $y'' \in Y$ iff $\Theta_w(y') \prec \Theta_w(y'')$ for all equitable rational preference relations $\preceq$, and that denoted by $y' \prec_{we} y''$.

Definition 2.4. We say that outcome vector $y \in Y$ is w-equitably nondominated iff there does not exit $y' \in Y$ such that $y' \prec_{we} y$. Also, we say that feasible solution $x \in X$ is a w-equitably efficient solution of the multiobjective problem (2.1), iff $y = f(x)$ is w-equitably nondominated.

Similar to the relation of w-equitable dominance, we can define the relation of w-equitable indifference $\simeq_{we}$ (indifference for all equitable rational preference relations) and the relation of w-equitable weak dominance $\preceq_{we}$ (weak preference for all equitable rational preference relations). To make it practical, w-equitable efficiency can be defined in terms of vector inequalities. In order to do that, we define certain mapping.

Definition 2.5. The cumulative ordering map $\overline{\Theta}_w : R^m_+ \to R^m_+$ is defined by,

$$\overline{\Theta}_w(y) = (\bar{\theta}_{w_1}(y), \bar{\theta}_{w_2}(y), \ldots, \bar{\theta}_{w_m}(y))$$

where $\bar{\theta}_{w_k}(y) = \sum_{j=1}^k w_j\theta_j(y)$ for $k = 1, 2, \ldots, m$ and θ_j's are as defined in Definition 2.2.

Note that in general $\overline{\Theta}(\Theta_w(y)) \neq \overline{\Theta}_w(y)$, but if $w_1 \geq w_2 \geq \ldots \geq w_m$ this relation is established.

Definition 2.6. Suppose that $y', y'' \in Y$ are two outcome vectors. The relation $\leq_{wi}$, $<_{wi}$ and $=_{wi}$ are defined as follows:

$$\begin{aligned} y' \leq_{wi} y'' &\Leftrightarrow \overline{\Theta}_w(y') \leqq \overline{\Theta}_w(y''), \\ y' <_{wi} y'' &\Leftrightarrow \overline{\Theta}_w(y') \leq \overline{\Theta}_w(y''), \\ y' =_{wi} y'' &\Leftrightarrow \overline{\Theta}_w(y') = \overline{\Theta}_w(y''). \end{aligned}$$

In the main theorem, we will discuss the relationship between two preference $\preceq_{we}$ and $\leq_{wi}$.

Theorem 2.7. *Let $w_1 \geq w_2 \geq \ldots \geq w_m$ and $y', y'' \in Y$ be two outcome vectors. We have*

$$\begin{aligned} y' \preceq_{we} y'' &\Leftrightarrow y' \leq_{wi} y'', \\ y' \prec_{we} y'' &\Leftrightarrow y' <_{wi} y''. \end{aligned}$$

Note that Theorem 2.7 permits one to express w-equitable efficiency for problem (2.1) in terms of the standard efficiency for the multiobjective problem with objectives $\overline{\Theta}_w(f(x))$:

$$\min\{\overline{\Theta}_w(f(x)) : x \in X\}. \tag{2.2}$$

Theorem 2.8. *Let $w_1 \geq w_2 \geq \ldots \geq w_m$ and let $x \in X$ be a feasible solution. x is efficient solution of the multiobjective problem (2.2) if and only if it is w-equitably efficient solution of the multiobjective problem (2.1).*

Remark 2.9. Note that if $w_j = 1$ for $j = 1, 2, \ldots, m$, then $\Theta_w(y) = \Theta(y)$ and $\overline{\Theta}_w(y) = \overline{\Theta}(y)$, so the relation $\preceq_{we}$ becomes the relation $\preceq_e$. Hence, the results of [1] are obtained.

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ON THE SOLUTION OF FUZZY OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, a novel methodology for detecting fuzzy optimal solutions of fuzzy optimal control problems governed by fuzzy differential equations is presented. For this purpose, using partial ordering and parametric representation, the discussed fuzzy optimal control problems are reduced to general optimal control problems in parametric forms. Then by Pontryagin's principle, a candidate for the solution of the original problem is derived.

1. Introduction

To deal with fuzzy numbers it suffices to operate with their α-level set. The use of α-level allows us to treat fuzzy numbers as a set of nested real intervals. There are several models to obtain parametric representations of fuzzy numbers and their arithmetic operators [1, 3]. Here, using the convex combinations of the α-level set bounds of fuzzy numbers, two parametric representations for α-level sets of fuzzy numbers are proposed. Based on these parametric representations, the parametric arithmetic for fuzzy numbers is defined, and the α-level set of fuzzy valued functions is expressed as a set of classical functions. Also, the concepts of derivative and integral for

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fuzzy valued functions are rewritten. Under these settings, the fuzzy optimal control problem converted to a general optimal control problem in the parametric form. Then, the Pontryagin's principle optimality condition can be naturally elicited, which its solution leads to the construction of α-level set of fuzzy optimal solution for the original problem.

2. Preliminaries

A fuzzy number A is a fuzzy set with non-empty bounded level sets $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$ for all $\alpha \in [0,1]$, where a_α^- and a_α^+ are its lower and upper bounds, respectively. We denote the class of fuzzy numbers by $\mathcal{F}(\mathbb{R})$. The α-level set $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$ allows us to consider non-decreasing and non-increasing parametric representations as follows:

$$[A]_\alpha = \{a(t,\alpha)|\ a(t,\alpha) = a_\alpha^- + t(a_\alpha^+ - a_\alpha^-);\ t \in [0,1]\}, \tag{2.1}$$

$$[A]_\alpha = \{a(t,\alpha)|\ a(t,\alpha) = a_\alpha^+ + t(a_\alpha^- - a_\alpha^+);\ t \in [0,1]\}. \tag{2.2}$$

If $[A]_\alpha = \{a(t,\alpha)|\ a(t,\alpha) = a_\alpha^- + t(a_\alpha^+ - a_\alpha^-);\ t \in [0,1]\}$ and $[B]_\alpha = \{b(t,\alpha)|\ b(t,\alpha) = b_\alpha^- + t(b_\alpha^+ - b_\alpha^-);\ t \in [0,1]\}$ are the non-decreasing representations of α-level sets of $A, B \in \mathcal{F}(\mathbb{R})$, respectively, and $\lambda \in \mathbb{R}$, then parametric arithmetics $A * B$, with $* \in \{+, -, \times, \div\}$, and $\lambda.A$ are defined as

$$[A * B]_\alpha = \{a(t_1,\alpha) * b(t_2,\alpha)|\ t_1, t_2 \in [0,1]\}, \tag{2.3}$$

$$[\lambda.A]_\alpha = \{\lambda a(t,\alpha)|\ t \in [0,1]\}. \tag{2.4}$$

The difference defined in (2.3) has the property $A - A \neq 0$. To overcome this situation, the following definition can be proposed.

Definition 2.1. [2] The parametric difference (p-difference for short) of two fuzzy numbers $A, B \in \mathcal{F}(\mathbb{R})$ is given by its α-level set as

$$[A \ominus_p B]_\alpha = \{a(t,\alpha) - b(t,\alpha)|\ a(t,\alpha) = a_\alpha^- + t(a_\alpha^+ - a_\alpha^-),\ b(t,\alpha) = b_\alpha^- + t(b_\alpha^+ - b_\alpha^-)\}.$$

If A and B be two fuzzy numbers, then

$$A \preceq B \Leftrightarrow a(t,\alpha) \leq b(t,\alpha),\ \forall\ t, \alpha \in [0,1]. \tag{2.5}$$

" $\preceq$ " is a partial ordering on fuzzy number space. Furthermore

$$A = B \Leftrightarrow a(t,\alpha) = b(t,\alpha),\ \forall\ t, \alpha \in [0,1]. \tag{2.6}$$

3. Fuzzy valued function

Let $\mathbf{C}^k = (C_1, C_2, \cdots, C_k)^T,\ C_j \in \mathcal{F}(\mathbb{R}),\ j = 1, \cdots, k$, be a k-dimensional fuzzy vector which each element is a fuzzy number, and denotes the set of all parameters that are present in a fuzzy valued function. Without loss of generality, consider $\mathbf{C}^k$ to be an ordered set with respect to (w.r.t.) the order maintained in the function. If the fuzzy valued function $F_{\mathbf{C}^k} : T \subseteq \mathbb{R} \to \mathcal{F}(\mathbb{R})$ is derived from a continuous function by applying Zadeh's extension principle, then using a non-decreasing parametric representation of α-level set of fuzzy numbers, the α-level set of fuzzy valued function can be expressed as a set of classical functions

$$[F_{\mathbf{C}^k}(x)]_\alpha = \left\{ f_{\mathbf{c}(\mathbf{t},\alpha)}(x) \middle| f_{\mathbf{c}(\mathbf{t},\alpha)} : T \subseteq \mathbb{R} \to \mathbb{R}; \mathbf{c}(\mathbf{t},\alpha) \in [\mathbf{C}^k]_\alpha \right\}.$$

Based on the concept of p-difference, the concept of p-differentiability of fuzzy valued function can be defined.

Definition 3.1. [2] Let $x_0 \in]a,b[$ and h be such that $x_0+h \in]a,b[$, then the p-derivative of the fuzzy valued function $F_{\mathbf{C}^k} :]a,b[\to \mathcal{F}(\mathbb{R})$ at x_0 is defined as

$$F'_{\mathbf{C}^k}(x_0) = \lim_{h\to 0} \frac{1}{h} \left[F_{\mathbf{C}^k}(x_0+h) \ominus_p F_{\mathbf{C}^k}(x_0)\right].$$

If $F'_{\mathbf{C}^k}(x_0) \in \mathcal{F}(\mathbb{R})$ exists, then $F_{\mathbf{C}^k}$ is called parametric differentiable (p-differentiable for short) at x_0.

Proposition 3.2. [2] *Let $F_{\boldsymbol{C}^k} :]a,b[\to \mathcal{F}(\mathbb{R})$ be defined in terms of its α-level set $[F_{\boldsymbol{C}^k}]_\alpha = \{f_{\boldsymbol{c}(\boldsymbol{t},\alpha)}(x) | f_{\boldsymbol{c}(\boldsymbol{t},\alpha)} :]a,b[\to \mathbb{R}, \boldsymbol{c}(\boldsymbol{t},\alpha) \in [\boldsymbol{C}^k]_\alpha\}$. If $f_{\boldsymbol{c}(\boldsymbol{t},\alpha)}(x)$ is differentiable at $x_0 \in]a,b[$ and for all $\alpha \in [0,1]$, $f_{\boldsymbol{c}(\boldsymbol{t},\alpha)}(x_0+h) - f_{\boldsymbol{c}(\boldsymbol{t},\alpha)}(x_0)$ satisfies the conditions of Negoita-Ralescu characterization theorem, then $F_{\boldsymbol{C}^k}$ is p-differentiable at x_0 and there exists $F'_{\boldsymbol{C}^k}(x_0) \in \mathcal{F}(\mathbb{R})$ such that*

$$[F'_{\boldsymbol{C}^k}(x_0)]_\alpha = \left\{ f'_{\boldsymbol{c}(\boldsymbol{t},\alpha)}(x_0) \,\middle|\, f_{\boldsymbol{c}(\boldsymbol{t},\alpha)} :]a,b[\to \mathbb{R}, \boldsymbol{c}(\boldsymbol{t},\alpha) \in [\boldsymbol{C}^k]_\alpha \right\}.$$

Moreover $[F'_{\boldsymbol{C}^k}(x_0)]_\alpha = \left[\min_{\boldsymbol{t}} f'_{\boldsymbol{c}(\boldsymbol{t},\alpha)}(x_0), \max_{\boldsymbol{t}} f'_{\boldsymbol{c}(\boldsymbol{t},\alpha)}(x_0)\right]$.

Definition 3.3. [2] The integral of a fuzzy valued function $F_{\mathbf{C}^k} : [a,b] \to \mathcal{F}(\mathbb{R})$ with $[F_{\mathbf{C}^k}(x)]_\alpha = \{f_{\mathbf{c}(\mathbf{t},\alpha)}(x) \mid f_{\mathbf{c}(\mathbf{t},\alpha)} : [a,b] \to \mathbb{R},\ \mathbf{c}(\mathbf{t},\alpha) \in [\mathbf{C}^k]_\alpha\}$ can be defined level-wise as

$$\left[\int_a^b F_{\mathbf{C}^k}(x)dx\right]_\alpha = \left\{ \int_a^b f_{\mathbf{c}(\mathbf{t},\alpha)}(x)dx \,\middle|\, f_{\mathbf{c}(\mathbf{t},\alpha)} : [a,b] \to \mathbb{R} \text{ is integrable w.r.t. } x \text{ for every} \mathbf{c}(\mathbf{t},\alpha) \in [\mathbf{C}^k]_\alpha \right\}.$$

Note that $\mathcal{C}(T,\mathcal{F}(\mathbb{R}))$ denotes the space of all continuous fuzzy valued functions $F_{\mathbf{C}^k} : T \subseteq \mathbb{R} \to \mathcal{F}(\mathbb{R})$, and $\hat{\mathcal{C}}^1(T,\mathcal{F}(\mathbb{R}))$ denotes the space of all continuously p-differentiable fuzzy valued functions $F_{\mathbf{C}^k} : T \subseteq \mathbb{R} \to \mathcal{F}(\mathbb{R})$ that their corresponding real valued function $f_{\mathbf{c}(\mathbf{t},\alpha)} : T \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable for all $\mathbf{t} \in [0,1]^k, \alpha \in [0,1]$.

4. Pontryagin's principle of optimality

Consider the fuzzy optimal control problem which is described as follows: Find the fuzzy control $U_{\mathbf{Z}^r} \in \mathcal{C}([x_0,x_f],\mathcal{F}(\mathbb{R}))$, and its corresponding fuzzy state $Y_{\mathbf{D}^n} \in \hat{\mathcal{C}}^1([x_0,x_f],\mathcal{F}(\mathbb{R}))$ such that the fuzzy pair $(Y_{\mathbf{D}^n}, U_{\mathbf{Z}^r})$ minimizes the fuzzy valued functional

$$J(Y_{\mathbf{D}^n}, U_{\mathbf{Z}^r}) = \int_{x_0}^{x_f} G_{\mathbf{C}^k}(x, Y_{\mathbf{D}^n}, U_{\mathbf{Z}^r})dx, \tag{4.1}$$

subject to

$$Y'_{\mathbf{D}^n} = H_{\mathbf{E}^m}(x, Y_{\mathbf{D}^n}, U_{\mathbf{Z}^r}) \tag{4.2}$$

with the following boundary conditions

$$Y_{\mathbf{D}^n}(x_0) = A,\ Y_{\mathbf{D}^n}(x_f) = B. \tag{4.3}$$

Definition 4.1. A fuzzy pair $(Y^*_{\mathbf{D}^n}, U^*_{\mathbf{Z}^r}) \in \hat{\mathcal{C}}^1([x_0, x_f], \mathcal{F}(\mathbb{R})) \times \mathcal{C}([x_0, x_f], \mathcal{F}(\mathbb{R}))$ is called an optimal solution for problem (4.1)-(4.3) with respect to partial order (2.5), if $Y^*_{\mathbf{D}^n}$ is the solution of (4.2)-(4.3) corresponding to the fuzzy optimal control $U^*_{\mathbf{Z}^r}$ and

$$J(Y^*_{\mathbf{D}^n}, U^*_{\mathbf{Z}^r}) \preceq J(Y_{\mathbf{D}^n}, U_{\mathbf{Z}^r}), \tag{4.4}$$

for every comparable pair $(Y_{\mathbf{D}^n}, U_{\mathbf{Z}^r}) \in \hat{\mathcal{C}}^1([x_0, x_f], \mathcal{F}(\mathbb{R})) \times \mathcal{C}([x_0, x_f], \mathcal{F}(\mathbb{R}))$ satisfying (4.2) and (4.3).

The next theorem give a necessary condition for the optimal solution of fuzzy optimal control problem (4.1)-(4.3) from Pontryagin's principle.

Theorem 4.2. *Let $G_{C^k}, H_{E^m} \in \hat{\mathcal{C}}^1([x_0, x_f], \mathcal{F}(\mathbb{R}), \mathcal{F}(\mathbb{R}))$ and a fuzzy pair $(Y^*_{\boldsymbol{D}^n}, U^*_{\boldsymbol{Z}^r})$ be an optimal solution for fuzzy optimal control problem (4.1)-(4.3). Then there exists a real valued function $\psi^*_{w(\boldsymbol{t},\boldsymbol{t}',\alpha)} \in C^1([x_0, x_f], \mathbb{R})$ such that*

$$\frac{\partial g_{\boldsymbol{c}(\boldsymbol{t}_1,\alpha)}}{\partial y_{\boldsymbol{d}(\boldsymbol{t},\alpha)}} + \psi^*_{w(\boldsymbol{t},\boldsymbol{t}',\alpha)} \frac{\partial h_{\boldsymbol{e}(\boldsymbol{t}_2,\alpha)}}{\partial y_{\boldsymbol{d}(\boldsymbol{t},\alpha)}} = -\psi^{*'}_{w(\boldsymbol{t},\boldsymbol{t}',\alpha)},$$
$$\frac{\partial g_{\boldsymbol{c}(\boldsymbol{t}_1,\alpha)}}{\partial u_{\boldsymbol{z}(\boldsymbol{t}',\alpha)}} + \psi^*_{w(\boldsymbol{t},\boldsymbol{t}',\alpha)} \frac{\partial h_{\boldsymbol{e}(\boldsymbol{t}_2,\alpha)}}{\partial u_{\boldsymbol{z}(\boldsymbol{t}',\alpha)}} = 0,$$

for all fixed $\boldsymbol{t} \in [0,1]^n, \boldsymbol{t}' \in [0,1]^r, \boldsymbol{t}_1 \in [0,1]^k, \boldsymbol{t}_2 \in [0,1]^m$ and $t_3, t_4, \alpha \in [0,1]$.

Hence, the α-level sets of the fuzzy optimal control $U^*_{\mathbf{Z}^r}$ and its corresponding fuzzy state $Y^*_{\mathbf{D}^n}$ for fuzzy optimal control problem (4.1)-(4.3), i.e.,

$$[U^*_{\mathbf{Z}^r}(x)]_\alpha = \{u^*_{\mathbf{z}(\mathbf{t}',\alpha)}(x) | u^*_{\mathbf{z}(\mathbf{t}',\alpha)} : [x_0, x_f] \to \mathbb{R}; \mathbf{z}(\mathbf{t}', \alpha) \in [\mathbf{Z}^r]_\alpha\},$$
$$[Y^*_{\mathbf{D}^n}(x)]_\alpha = \{y^*_{\mathbf{d}(\mathbf{t},\alpha)}(x) | y^*_{\mathbf{d}(\mathbf{t},\alpha)} : [x_0, x_f] \to \mathbb{R}; \mathbf{d}(\mathbf{t}, \alpha) \in [\mathbf{D}^n]_\alpha\},$$

are obtained from optimal solution of its corresponding general optimal control problem in the parametric form

$$\min\ j(y_{\mathbf{d}(\mathbf{t},\alpha)}, u_{\mathbf{z}(\mathbf{t}',\alpha)}) = \int_{x_0}^{x_f} g_{\mathbf{c}(\mathbf{t}_1,\alpha)}(x, y_{\mathbf{d}(\mathbf{t},\alpha)}, u_{\mathbf{z}(\mathbf{t}',\alpha)}) dx,$$
$$s.t.\ y'_{\mathbf{d}(\mathbf{t},\alpha)} = h_{\mathbf{e}(\mathbf{t}_2,\alpha)}(x, y_{\mathbf{d}(\mathbf{t},\alpha)}, u_{\mathbf{z}(\mathbf{t}',\alpha)}),$$
$$y_{\mathbf{d}(\mathbf{t},\alpha)}(x_0) = a(t_3, \alpha),\ y_{\mathbf{d}(\mathbf{t},\alpha)}(x_f) = b(t_4, \alpha).$$

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INFINITE ORDER INDUCED BY w-EQUITABLE DOMINANCE FOR MULTIOBJECTIVE OPTIMIZATION PROBLEMS

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Abstract. In this paper, the concept of infinite order w-equitable dominance is introduced as a refinement of equitable and w-equitable dominance concepts. Moreover an algorithm is presented to generate subsets of equitably efficient solutions, which aims to offer a limited number of representative solutions to the decision maker.

1. Introduction

In equitable multiobjective optimization, the focus is on the distribution of outcome values while ignoring their ordering. This means that, we are interested in a set of values of the objectives without taking into account which objective is taking a specific value. Kostreva et al. [1, 2] are the first ones who introduced the concept of equitability into the multiobjective programming. They have shown that the set of equitably efficient solutions are contained within the set of efficient solutions for the same problem. More recently, the authors [3]

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generalized the concept of equitable efficiency by introducing equitable B-efficiency.

Throughout this article the following notation is used. Let R^m be the Euclidean vector space and $y', y'' \in R^m$. $y' \leqq y''$ denotes $y'_i \leq y''_i$ for all $i = 1, \ldots, m$. $y' < y''$ denotes $y'_i < y''_i$ for all $i = 1, \ldots, m$. $y' \leq y''$ denotes $y' \leqq y''$ but $y' \neq y''$. The set $\{y \in R^m : y' \geqq 0\}$ is denoted by R^m_+. Consider an optimization problem

$$\begin{aligned} &\min \{(f_1(x), f_2(x), \ldots, f_m(x))\}, \\ &\textit{subject to } x \in X \end{aligned} \tag{1.1}$$

where x denotes a vector of decision variables selected from the feasible set X and $f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ is a vector function that maps the feasible set X into the objective (criterion) space R^m_+ and $Y = f(X)$.

2. Main results

The following definition is a necessary notion for the solution concepts of interest in this paper.

Definition 2.1. Let $y \in R^m_+$.

1. An ordering map $\Theta : R^m_+ \to R^m_+$ is a function defined as $\Theta(y) = (\theta_1(y), \theta_2(y), \ldots, \theta_m(y))$, where $\theta_1(y) \geq \theta_2(y) \geq \ldots \geq \theta_m(y)$, $\theta_i(y) = y_{\tau(i)}$ for $i = 1, 2, \ldots, m$, and τ is a permutation of the set $\{1, 2, \ldots, m\}$.
2. Let $w = (w_j)_{j=1}^m$ be a finite sequence and $w_1 \geq w_2 \geq \ldots \geq w_m$. The ordering map $\Theta_w : R^m_+ \to R^m_+$ is defined by, $\Theta_w(y) = (w_1\theta_1(y), w_2\theta_2(y), \ldots, w_m\theta_m(y))$. Also the cumulative ordering map $\overline{\Theta}_w : R^m_+ \to R^m_+$ is defined by, $\overline{\Theta}_w(y) = (\bar{\theta}_{w_1}(y), \bar{\theta}_{w_2}(y), \ldots, \bar{\theta}_{w_m}(y))$ where $\bar{\theta}_{w_k}(y) = \sum_{j=1}^k w_j\theta_j(y)$ for $k = 1, 2, \ldots, m$.

Definition 2.2. We say that a vector $y' \in Y$ w-equitably dominates $y'' \in Y$, denoted by $y' \leq_{we} y''$, if and only if

$$\overline{\Theta}_w(y') \leq \overline{\Theta}_w(y'').$$

A vector $y \in Y$ is w-equitably nondominated iff there does not exit $y' \in Y$ such that $y' \leq_{we} y$. Also, we say that the feasible solution $x \in X$ is a w-equitably efficient solution of the multiobjective problem (1.1), iff $y = f(x)$ is w-equitably nondominated.

Theorem 2.3. *Let $w = (w_j)$ and $v = (v_j)$ be two vectors in R^m_+. If $\frac{w_1}{v_1} \geq \frac{w_2}{v_2} \geq \ldots \geq \frac{w_m}{v_m}$, then $y' \leq_{vi} y'' \Rightarrow y' \leq_{wi} y''$ for all $y', y'' \in R^m_+$. This means that, if $x \in X$ is a w-equitably efficient solution of the multiobjective problem(1.1), then it is a v-equitably efficient solution of the same problem.*

From now on the set of nonnegative integers is denoted by $\mathbb{N} = \{0, 1, 2, \ldots\}$. If $k \in \mathbb{N}$, we put $w^k = (w_i^k)$, especially for $k = 0$, $w^0 = (1, 1, \ldots, 1)$ and $\leq_{w^0 i} = \prec_e$.

Corollary 2.4. *If $w = (w_j)$ is a decreasing sequence in R_+^m then*

$$y' \leq_{w^k i} y'' \Rightarrow y' \leq_{w^{k+1} i} y''$$

for all $y', y'' \in R_+^m$ and $k \in \mathbb{N}$. This means that, if $x \in X$ is a w^{k+1}-equitably efficient solution of the multiobjective problem(1.1), then it is a w^k-equitably efficient solution of the same problem.

We define infinite order w-equitable dominance as follows: $\leq_{w^\infty} = \bigcup_{k \in \mathbb{N}} \leq_{w^k}$. This means that,

$$y' \leq_{w^\infty} y'' \Leftrightarrow y' \leq_{w^k} y'' \quad (for\ some\ k \in \mathbb{N}).$$

For example suppose that $y' = (6, 1, 4), y'' = (2, 7, 0.5), w_1 = 0.8, w_2 = 0.6$ and $w_3 = 0.4$. Since $\overline{\Theta}_{w^k}(y')$ and $\overline{\Theta}_{w^k}(y'')$ cannot be compared in terms of w^k-equitable dominance for $k = 0, 1, 2$, we compare instead $\overline{\Theta}_{w^3}(y')$ and $\overline{\Theta}_{w^3}(y'')$. We have $\overline{\Theta}_{w^3}(y') = (3.072, 3.936, 4)$ and $\overline{\Theta}_{w^3}(y'') = (3.584, 4.016, 4.048)$, so $y' \leq_{w^3} y''$. Hence $y' \leq_{w^\infty} y''$.
By interaction with Corollary 2.4, we offer an algorithm to reduce the equitably efficient solutions of the multiobjective problem (1.1).

Algorithm 2.5. *Step 1. Put $k = 0$; Step 2. Get weights $w_1, w_2, \ldots, w_m$ such that $w_1 \geq w_2 \geq \ldots \geq w_m$, according to the decision maker; Step 3. Put $w = w^k$; Step 4. Solve the multiobjective problem* $\min\{\overline{\Theta}_w(f(x)) : x \in X\}$*; Step 5. If the decision maker chooses the desired solution, stop; Otherwise, put $k = k + 1$ and go to Step 3.*

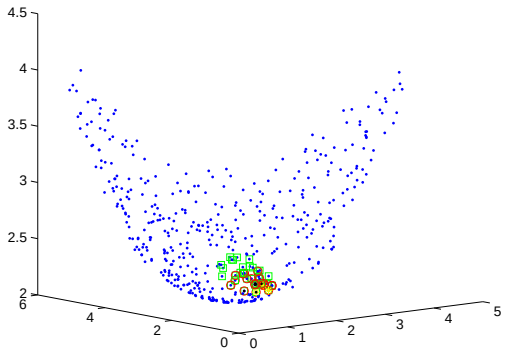

Figure 1: The Pareto, equitable and equitable w-dominance fronts (objective space) for the VFM1 problem (2 variables and 3 objectives).

In the following example, a large number of random solutions are generated for scalable test function. From this large set of solutions, the nondominated solutions with respect to Pareto dominance, equitable dominance and w-equitable dominance are calculated.

Example 2.6. The test problem considered is the VFM1 [4],

$$\begin{aligned} \min_{x\in R^2} \quad & y = \{f_1(x), f_2(x), f_3(x)\} \\ & f_1(x) = x_1^2 + (x_2 - 1)^2 \\ & f_2(x) = x_1^2 + (x_2 + 1)^2 + 1 \\ & f_3(x) = (x_1 - 1)^2 + x_2^2 + 2 \\ & x_1, x_2 \in [-2, 2]. \end{aligned}$$

Figure 1 shows the Pareto, equitable and w-equitable dominance fronts (objective space) for $w_1 = 0.8, w_2 = 0.6$ and $w_3 = 0.4$. In Figure 1 from 5000 random solutions, 469 solutions (blue point) are rationally nondominated. Suppose that $w_1 = 0.8, w_2 = 0.6$ and $w_3 = 0.4$, 39 solutions (green diamond) are equitably nondominated which are obtained in first iteration of Algorithm 2.5. In second, third and fourth iteration of Algorithm 2.5 24 solutions (red circle), 12 solutions (black point) and 3 solutions (yellow star) are obtained which are w^k-equitably nondominated for $k = 1, 2, 3$, respectively.

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DETERMINING MINIMUM SET OF MASTER REGULATORY GENES BY A PARAMETRIC ALGORITHM

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Abstract. Identification of master regulatory genes is an important task in systems biology. The aim of this paper is to determine minimum cardinality set of master regulatory genes with more influence in weighted gene regulatory network. This problem can be formulated as integer biobjective problem based on the minimum dominating set model. The biobjective problem is reduced to single objective by sum-weighted method, then a parametric algorithm is introduced for solving the single objective problem. The proposed model is applied on a human network.

1. Introduction

Gene regulatory networks (GRN) are of well-known biological networks that consist of a set of genes or proteins, and regulatory relationships between them. The analyzing of these networks is very important in drug design for various lethal diseases such as cancer.

There is set of genes known as master regulatory genes that involved in controlling the expression of other genes in a gene regulatory network. In this paper, we consider weighted gene regulatory network and determine a set of master regulatory genes with minimum cardinality and maximum weight. This problem is formulated as biobjective optimization problem based on the minimum dominating set problem. Firstly, master regulatory gene sets with minimum cardinality is determined, then a set is identified which has maximum weight.

Key words and phrases. Dominating set, Biology, Master gene, Gene network.
* Speaker.

The paper is organized as follows: In the next section, we state and formulate the minimum dominating set problem. In Section 3, we present the integer biobjective programming formulation of the weighted gene regulatory network. The computational results are given in Section 4, and Section 5 contains the conclusions.

2. Minimum dominating set

A dominating set (DS) for graph $G = (V, E)$ is a subset of vertices $S \subseteq V$, where every vertex $v \in V$ is either $v \in S$ or is adjacent to at least one vertex in S. A dominating set $S \subseteq V$ is the minimum dominating set (MDS) if it has the smallest cardinality dominating set among all dominating sets of the graph G. It is well-known that finding a minimum dominating set in a graph G is an $\mathcal{NP}$-hard decision [2]. In MDS problem, we assign a binary variable x_v associated to each node $v \in V$ in graph $G = (V, E)$. The MDS problem formulated as integer linear programming:

$$\min \Big\{ \sum_{v \in V} x_v \;\Big|\; x_v + \sum_{(u,v) \in E} x_u \geq 1,\ x_v \in \{0, 1\},\ \forall v \in V \Big\}. \tag{2.1}$$

If $x_v = 1$, then node v belongs to MDS. There may exist more than one optimal solution for problem (2.1). Therefore, it is difficult to find which MDS represents the more useful nodes. To overcome this problem, in the Section 3, we will introduce weighted gene regulatory network problem. In gene network each gene has a collective influence, in this paper we assume collective influence of each gene as its weight.

Collective Influence. Collective influence (CI) describes how many other nodes can be reached from a given node [3]. The collective influence of a node v is defined as the following formula:

$$\mathrm{CI}_\ell(v) = (d_v - 1) \sum_{u \in \partial \mathrm{Ball}(v,\ell)} (d_u - 1) \tag{2.2}$$

where d_v is the degree of node v and $\partial \mathrm{Ball}(v, l)$ is the set of nodes at distance ℓ from node v. There exists a free parameter ℓ, the distance, which according to [3], we choose $\ell \geq 1$ but not too large (e.g. $\ell = 1, 2, 3$), because in too large distance ℓ the boundaries of the network are reached and the collective influence of all nodes approaches zero.

3. The weighted gene regulatory network formulation

Consider a given WGRN is represented as a graph $G = (V, E, w)$. A node $v \in V$ corresponds to a gene of WGRN and an edge $(u, v) \in E$ represents an interaction between two genes u and v. The function $w : V \to \mathbb{R}^+$ associates a nonnegative weight to each gene of the set

V. Here, weight w_v shows the relative collective influence value of node v. In presented model, the task is to identify a set of genes with minimum cardinality and maximum weight in a WGRN, such that set of determined master genes control the given WGRN. Therefore, the problem formulated as following biobjective program

$$\begin{aligned} \text{WGRN}: \ \min \sum_{v\in V} x_v, \ \max \sum_{v\in V} w_v x_v & \qquad (3.1)\\ \text{s.t. } x_v + \sum_{(u,v)\in E} x_u \geq 1, & \qquad \forall v \in V,\\ x_v \in \{0,1\}, & \qquad \forall v \in V. \end{aligned}$$

3.1. **Solving problem (3.1) by parametric algorithm.** The WGRN problem can be reduced to a single objective problem by using weighted-sum method. The scalarized relaxation optimization problem with parametric weight $\alpha \in (0,1)$ is as follows

$$g(\alpha) = \min \ \big\{(\frac{1}{w} + \alpha(1-\frac{1}{w}))^T x \ \big| \ (A+I)x - r = 1, \ x, r \geq 0\big\},$$

where $A \in \mathbb{R}^{n\times n}$ is adjacency matrix and $I \in \mathbb{R}^{n\times n}$ is identical matrix and vector $r \in \mathbb{R}^n$ is slack vector.

Suppose $c = \frac{1}{w}$, $\Delta c = 1 - \frac{1}{w}$, $c(\alpha) = c + \alpha\Delta c$. Algorithm 1 calculated the optimal value function $g(\alpha)$ when the vector c is perturbed by a scalar multiple of Δc to $c(\alpha) = c + \alpha\Delta c$ [4].

Algorithm 1 The Optimal Value Function $g(\alpha)$, $\alpha \geq 0$

Input
An optimal solution (x^*, r^*) of (P_α) with $\alpha = 0$ and a perturbation vector Δc;
Begin
ready=false, $k = 1$, $x^0 = x^*$, $r^0 = r^*$;
while not ready **do**
  Solve $\max_{\alpha,y,s}\{\alpha : A^T y + s = c + \alpha\Delta c, \ s^T x^{k-1} = 0, \ y^T r^{k-1} = 0, \ s, y \geq 0,\}$;
  if this problem is unbounded **then**
    ready=true;
  else
    let (α_k, y^k, s^k) be an optimal solution;
  end if
  Solve $\min\limits_{x,r}\{\Delta c^T x : Ax - r = b, \ x^T s^k = 0, \ r^T y^k = 0, \ x, r \geq 0\}$;
  if this problem is unbounded **then**
    ready=true;
  else
    let (x^k, r^k) be an optimal solution, $k = k + 1$;
  end if
end while
End

The algorithm 1 finds the successive break points of function g on the nonnegative real line as well as the slopes of g on the successive linearity intervals. Here, we solved all the programming problems by using solver CPLEX that can solve large-scale programming problems.

4. Computational experience

In this section, WGRN model is implemented on a human network with binary interactions for H. sapiens. The dataset H. sapiens was taken from the High-quality INTeractomes (HINT) database [1]. For solving WGRN model, first weighted network is constructed by computing a weight for each gene of given network using definition CI. Table 1 shows information and computational results for H. sapiens network by Algorithm 1. By using computational results, one can see that what percentage of genes can control entire given network.

Table 1. Statistics and the number of master regulatory genes of binary dataset H. sapiens.

Number of Genes	Number of Interactions	α	Number of master genes	% Master Genes
7,865	24,368	0	1412	17.95
		2.2182e-05	1391	17.69
		1	1391	17.69

5. Conclusion

In this study, the weighted gene regulatory network was introduced by appropriation a weight for each gene. The proposed model formalized the problem of identifying a set master regulatory genes with the minimum cardinality set and maximum weight as integer biobjective problem. This model is reduced to single objective problem by using weighted-sum. Then this problem is solved by a parametric algorithm. Finally, the proposed model was applied on a human network.

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AN ANALYTICAL APPROACH FOR SOLVING FRACTIONAL FUZZY OPTIMAL CONTROL PROBLEM WITH FUZZY INITIAL CONDITIONS

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Abstract. The aim of this paper, is to introduce an analytically solution for fractional fuzzy optimal control Bolza problems when the initial state is also fuzzy. First the problem is transformed into two fractional optimal control problems by concept of α-cut. Then, we apply a new Riccati differential equation determined from Pontryagin's minimum principle to transfer each mentioned fractional optimal control problem to a fractional differential system. By showing the existence of solution, numerical simulation is also presented for different values of fractional order and the results are compared.

1. Introduction

We know that fuzziness is a very adequate tool to present many phenomena; also, in resent years, fractional calculus plays a very important roles in mathematics and other subject. Both these two facts together cause to face with the fuzzy fractional problem in optimal control theory where a fuzzy differential system in fractional order (FFOCP) is involved [2]. To continue, here, we consider a fractional fuzzy control

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* Speaker.

problem and turn it to two fractional control problems and then we use the fractional Pontryagin Maximum Principle to solve it. Consider the following optimal control problem:

$$Min : \int_a^b f_0(t, \tilde{x}(t), \tilde{u}(t))dt$$

$$S.to : \left(D^{\beta}_{\alpha^+}\tilde{x}\right)(t) = f_0(t, \tilde{x}(t), \tilde{u}(t)); \tilde{x}(a) = \tilde{x}_0, \tag{1.1}$$

$$\tag{1.2}$$

where $t \in (a, b) \subseteq \mathbb{R}$, $\tilde{x}$ is a fuzzy bounded trajectory, $\tilde{x}_0$ is a fuzzy initial condition, $\tilde{u}$ is fuzzy control variable, f and f_0 are two given continues functions [4]; here $\left(D^{\beta}_{a^+}\tilde{x}\right)(t)$ denotes the left Riemann-Liouville derivative at order $\beta \in (0, 1)$. We remind that the aim of this paper, is to find a fuzzy solution for these kind of problems.

2. Fuzzy Riemann-liouville differential

This section, is devoted to present the definition of fuzzy Riemann-Liouville integrals and derivatives by Hukuhara difference. We denote $C^F[a, b]$ as the space of all continuous fuzzy-valued functions on $[a, b]$ [2].

Definition 2.1. Let $f \in C^F[a, b] \cap L^F[a, b]$. The α-cut representation of f is shown by $f(x; \alpha) = \left[\underline{f}(x; \alpha), \bar{f}(x; \alpha)\right]$ for $0 \leq \alpha \leq 1$, where $\underline{f}(x; \alpha)$ and $\bar{f}(x; \alpha)$ are defined as lower and upper bounds of α-level set of f. Moreover, for $x_0 \in (a, b) \subseteq \mathbb{R}$ and $\Phi(x) \equiv \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f(t)}{(x-t)^{(\beta)}}$, f is called Riemann-Liouville H-differentiable of order $0 < \beta \leq 1$ at x_0, if there exist an element $\left(D^{\beta}_{(\alpha+)}f\right)(x_0)$ such that for sufficiently small $h > 0$, we have ([19]) $\left(D^{\beta}_{(\alpha+)}f\right)(x_0) = \lim_{h\to 0^+} \frac{\Phi(x_0+h) \ominus \Phi(x_0)}{h} = \lim_{h\to 0^+} \frac{\Phi(x_0) \ominus \Phi(x_0-h)}{h}$ where $\ominus$ is the Hukuhara difference [1]; also, we have the similar results for $\left(D^{\beta}_{(b^-)}f\right)(x\,; a)$.

3. Fractional Fuzzy Optimal Control Problem

Consider the following FFOCPs:

$$Min : \int_a^b f_0(t, \tilde{x}(t), \tilde{u}(t))dt$$

$$S.\ to : \left(D^{\beta}_{\alpha+}\tilde{x}\right)(t) = f(t, \tilde{x}(t), \tilde{u}(t)); \tilde{x}(a) = \tilde{x}_0 = (p, q, r), \tag{3.1}$$

where the initial condition, $\tilde{x}_0 = (p, q, r)$ is a triangular fuzzy number and $0 < \beta \leq 1$. By using the concept of α-cut and parameterization of a fuzzy number [3], for each $0 \leq \alpha \leq 1$ we can write (3.1) in complex space as follows:

$$Min : \int_a^b f_0 \left(t, \underline{x}(t;\alpha), \underline{u}(t;\alpha) \right) + i f_0 \left(t, \bar{x}(t;\alpha), \bar{u}(t;\alpha) \right) dt$$

$$S.\ to : \begin{cases} \left(D^{\beta}_{(\alpha+)} \underline{x} \right)(t;\alpha) + i \left(D^{\beta}_{(\alpha+)} \bar{x} \right)(t;\alpha) = \\ f \left(t, \underline{x}(t;\alpha), \underline{u}(t;\alpha) \right) + i f \left(t, \bar{x}(t;\alpha), \bar{u}(t;\alpha) \right); \\ \underline{x}(a;\alpha) + i \bar{x}(a;\alpha) = (q\alpha + p(1-\alpha)) + i \left(q\alpha + r(1-\alpha) \right); \end{cases}$$

hence, the new description of (3) can be turned in (3.2) and (3.3):

$$Min : \int_a^b f_0 \left(t, \underline{x}(t;\alpha), \underline{u}(t;\alpha) \right) dt$$

$$S.\ to : \begin{cases} \left(D^{\beta}_{(\alpha+)} \underline{x} \right)(t;\alpha) = f \left(t, \underline{x}(t;\alpha), \underline{u}(t;\alpha) \right); \\ \underline{x}(a;\alpha) = (q\alpha + p(1-\alpha)); \end{cases} \tag{3.2}$$

and

$$Min : \int_a^b f_0 \left(t, \bar{x}(t;\alpha), \bar{u}(t;\alpha) \right) dt$$

$$S.\ to : \begin{cases} \left(D^{\beta}_{(\alpha+)} \bar{x} \right)(t;\alpha) = f \left(t, \bar{x}(t;\alpha), \bar{u}(t;\alpha) \right); \\ \bar{x}(a;\alpha) = (q\alpha + r(1-\alpha)); \end{cases} \tag{3.3}$$

Solving them, for given α we generate the optimal pairs $\left(\underline{x}^*(t;\alpha), \underline{u}^*(t;\alpha) \right)$ and $(\bar{x}^*(t;\alpha), \bar{u}^*(t;\alpha))$ for (3.2) and (3.3) respectively; therefore, solution of (3.1) is determined as:

$$\tilde{x}^*(t,\alpha) = \left[\underline{x}^*(t;\alpha), \bar{x}^*(t;\alpha) \right]; \tilde{u}^*(t,\alpha) = \left[\underline{u}^*(t;\alpha), \bar{u}^*(t;\alpha) \right]$$

4. Riccati differential equation for Fractional OCP

For the fractional optimal control problem:

$$min\ J = \frac{1}{2} S(t_1) x^2(t_1) + \frac{1}{2} \int_{t_0}^{t_1} \left\{ P(t) x^2(t) + 2q(t) x(t) u(t) + r(t) u^2(t) \right\} dt \tag{4.1}$$

$$S.\ to: \left(D_{t_0}^{\beta} x\right)(t) = a(t)x(t) + b(t)u(t); x(t_0) = x_0. \tag{4.2}$$

The Hamiltonian can be shown as:

$$H = \frac{1}{2}P(t)x^2 + q(t)x(t)u(t) + \frac{1}{2}r(t)u^2(t) + \lambda(a(t)x(t) + b(t)u(t)). \tag{4.3}$$

Then, based on the fractional Pontryagin system in (4.2), we have:

$$\left(D_{t_0}^{\beta}\lambda\right)(t) = \frac{\partial H}{\partial x} = px + qu + \lambda a; \tag{4.4}$$

$$\frac{\partial H}{\partial u} = qx + ru + \lambda a = 0. \tag{4.5}$$

(4.5) gives us:

$$u = -r^{-1}(qx - \lambda b). \tag{4.6}$$

By applying (4.6) in the differential equation (4.2), we have:

$$D_{t_0}^{\beta} x = (a - r^{-1}q)x - r^{-1}\lambda b^2. \tag{4.7}$$

In the similar way, we obtain:

$$D_{t_1}^{\beta}\lambda = (p - r^{-1}q^2)x + (-r^{-1}qb + a)\lambda. \tag{4.8}$$

Now, let $\lambda(t) = k(t)x(t)$; then we can reach to:

$$\left(D_{t_1}^{\beta} k\right)(t) = (p - r^{-1}q^2)x + (-r^{-1}qb + a)k(t); k(t_1) = s(t_1). \tag{4.9}$$

Then from calculating $k(t)$, one can determine $\lambda(t)$ according to $x(t)$. This fact makes (4.7) a fractional differential equation with an initial condition; by solving it we can compute the optimal trajectory for (4.2); then (4.6) gives us the optimal control of the original problem.

Regarding the shortage of space, numerical results will be presented in the oral speech.

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LINEAR APPROXIMATION AND GAP FUNCTION FOR OPTIMIZATION PROBLEMS WITH NONDIFFERENTIAL QUASICONVEX DATA

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Abstract. The purpose of this paper is to give a linear characterization and also a gap function for quasiconvex programming problems using adapted subdifferentials which generalize known results in convex case.

1. Introduction and Preliminaries

In the present paper we consider an optimization problem, defined as follows:

$$(\mathcal{M}): \text{ minimize } f(x) \text{ subject to } x \in C := \big\{x \in \mathbb{X} \mid g_1(x) \leq 0, \ldots, g_n(x) \leq 0\big\},$$

where f and g_i for $i = 1, \ldots, n$ are functions defined on a normed vector space $\mathbb{X}$ with values in $\mathbb{R} \cup \{\infty\}$.

In the field of nonsmooth optimization, generalized convex functions play an interesting and important role because of their theoretical aspects as well as their wide range of applications. In this article, as an application of the results given in Ref. [4], a connection between linear programs and nonlinear quasiconvex problems is given. Considering a feasible solution x_0 of $(\mathcal{M})$, a linear approximation of $(\mathcal{M})$, called

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* Speaker.

$(\mathcal{LM})$, is constructed and relationship between the optimality of x_0 for $(\mathcal{M})$ and its optimality for $(\mathcal{LM})$ is investigated. The final part of this note is devoted to characterizing the optimality for $(\mathcal{M})$ utilizing the gap function notion. This notion is a very important tool in studying optimization problems and variational inequalities; see e.g. [1, 2, 3].

We observe that since the proofs of our theorems are long, we did not state them in this extended abstract.

Hereafter, $\varphi : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ is a function which is finite at some point $x_0 \in \varphi^{-1}(\mathbb{R})$. The Plastria subdifferential and the Gutiérrez subdifferential of φ at x_0 are respectively defined as

$$\partial^{<}\varphi(x_0) := \big\{x^* \in \mathbb{X}^* \mid \varphi(x)-\varphi(x_0) \geq \big\langle x^*, x-x_0\big\rangle \quad \forall x \in [\varphi < \varphi(x_0)]\big\},$$

$$\partial^{\leq}\varphi(x_0) := \big\{x^* \in \mathbb{X}^* \mid \varphi(x)-\varphi(x_0) \geq \big\langle x^*, x-x_0\big\rangle \quad \forall x \in [\varphi \leq \varphi(x_0)]\big\},$$

where $\mathbb{X}^*$ denotes the topological dual space of $\mathbb{X}$, and

$$[\varphi < \varphi(x_0)] := \big\{x \in \mathbb{X} \mid \varphi(x) < \varphi(x_0)\big\},$$

$$[\varphi \leq \varphi(x_0)] := \big\{x \in \mathbb{X} \mid \varphi(x) \leq \varphi(x_0)\big\}.$$

Some basic calculus rules of these subdifferentials can be found in Ref. [5]. In Particular, both $\partial^{<}\varphi(x_0)$ and $\partial^{\leq}\varphi(x_0)$ are either empty or unbounded, and $\partial^{\leq}\varphi(x_0) \subseteq \partial^{<}\varphi(x_0)$. Also, $\partial^{<}\varphi(x_0) = \mathbb{X}^*$ iff $0 \in \partial^{<}\varphi(x_0)$ iff $0 \in \partial^{\leq}\varphi(x_0)$ iff x_0 is a minimizer of φ.

We say that $\varphi : \mathbb{X} \to \mathbb{R} \cup \{\infty\}$ is a Plastria function at x_0 if its strict sublevel set $[\varphi < \varphi(x_0)]$ is convex and such that

$$N\Big([\varphi < \varphi(x_0)], x_0\Big) = \mathbb{R}_+ \partial^{<}\varphi(x_0),$$

where $N(M, \bar{x})$ denotes the normal cone of convex set $M \subseteq \mathbb{X}$ at $\bar{x} \in \mathbb{X}$, i.e.,

$$N(M, \bar{x}) := \big\{y^* \in \mathbb{X}^* \mid \big\langle y^*, x - x_0\big\rangle \leq 0 \quad \forall x \in M\big\}.$$

We also say φ is a Gutiérrez function at x_0 if its sublevel set $[\varphi \leq \varphi(x_0)]$ is convex and such that

$$N\Big([\varphi \leq \varphi(x_0)], x_0\Big) = \mathbb{R}_+ \partial^{\leq}\varphi(x_0).$$

The Plastria and Gutiérrez functions have introduced in Ref. [4]. There are characterized some various classes of functions which are being Plastria and\or Gutiérrez.

2. Main Results

At starting point of this section, for each $i \in \{1, \ldots, n\}$, set

$$C_i := \big\{x \in \mathbb{X} \mid g_i(x) \leq 0\big\}.$$

Let $x_0 \in C$ be a feasible solution to $(\mathcal{M})$. Consider the following linear programming problem $(\mathcal{LM}_{x_0}^{x_0^*})$, for some $x_0^* \in \partial^< f(x_0)$

$$(\mathcal{LM}_{x_0}^{x_0^*}): \qquad \text{minimize } f_{x_0}^{x_0^*}(x) := f(x_0) + \langle x_0^*, x - x_0\rangle \;\; s.t. \;\; x \in C_{x_0},$$

in which

$$C_{x_0} = \Big\{x \in \mathbb{X} \mid \langle x_i^*, x - x_0\rangle \leq 0, \;\forall i \in \{1, \ldots, n\}, \forall x_i^* \in \partial^{\leq} g_i(x_0)\Big\}.$$

The above model is a linear infinite programming problem, because of the number of its constraints. The following theorem establishes the relationship between the optimality of x_0 for $(\mathcal{M})$ and $(\mathcal{LM}_{x_0}^{x_0^*})$.

Theorem 2.1. *Let x_0 be a solution to $(\mathcal{M})$ which is not a local minimizer of f. Let $I := \big\{i \in \{1, \ldots, n\} \mid g_i(x_0) = 0\big\}$. Assume that f is a Plastria function at x_0, $g_1, \ldots, g_n$ are u.s.c. at x_0 and that for every $i \in I$, g_i is a Gutiérrez functions at x_0. Furthermore, assume that, there exist some $k \in I$ and some $z \in C_k$ such that $g_i(z) < 0$ for each $i \in I\backslash\{k\}$ (Slater condition). Then x_0 is an optimal solution to $(\mathcal{LM}_{x_0}^{x_0^*})$ for some $x_0^* \in \partial^< f(x_0)$.*

The above theorem is valid if one replaces C_{x_0} with the bigger set

$$\widehat{C}_{x_0} := \Big\{x \in \mathbb{X} \mid g_i(x_0) + \langle x_i^*, x - x_0\rangle \leq 0, \;\forall i \in I, \forall x_i^* \in \partial^{\leq} g_i(x_0)\Big\} =$$

$$\Big\{x \in \mathbb{X} \mid \langle x_i^*, x - x_0\rangle \leq 0, \;\forall i \in I, \forall x_i^* \in \partial^{\leq} g_i(x_0)\Big\}.$$

The converse of Theorem 2.1 is not valid. Example 2.2 shows it. But if one considers $\widehat{C}_{x_0}$ instead of C_{x_0}, then the converse holds as well. Theorem 2.3 addresses this result.

Example 2.2. Consider the following problem:

$$\begin{array}{ll} \min & f(x) := x \\ s.t. & g_1(x) \leq 0, \\ & g_2(x) \leq 0, \end{array}$$

where

$$g_1(x) := \begin{cases} 2x & x \leq 0 \\ x & x > 0. \end{cases} \qquad \text{and} \qquad g_2(x) := -x - 1.$$

Here, $x \in \mathbb{R}$, and $C = [-1, 0]$. Considering $x_0 = 0$, we have

$$\partial^< f(x_0) = [1, +\infty), \quad \partial^{\leq} g_1(x_0) = [2, +\infty), \quad \partial^{\leq} g_2(x_0) = (-\infty, -1].$$

Hence, $(\mathcal{LM}_{x_0}^{x_0^*})$ can be written as follows

$$\begin{array}{lll} \min & \alpha x & \\ s.t. & \beta x \leq 0, & \forall \beta \geq 2, \\ & -1+\theta x \leq 0, & \forall \theta \leq -1, \end{array}$$

in which $\alpha \geq 1$ is a fixed scalar. The only feasible solution of this problem is $x_0 = 0$. Therefore, x_0 is the optimal solution of $(\mathcal{LM}_{x_0}^{x_0^*})$ while it is not optimal for the considered $(\mathcal{M})$.

Theorem 2.3. *Let $x_0 \in C$ be an optimal solution to*

$$(\widehat{\mathcal{LM}_{x_0}^{x_0^*}}): \qquad \text{minimize } f_{x_0}^{x_0^*}(x) := f(x_0) + \langle x_0^*, x - x_0 \rangle \quad s.t. \quad x \in \widehat{C}_{x_0},$$

for some $x_0^* \in \partial^< f(x_0)$. If f is finite at x_0, then x_0 is an optimal solution to $(\mathcal{M})$.

Now, along the lines of [1], the gap function $\phi : \mathbb{X} \times \mathbb{X}^* \longrightarrow \mathbb{R}$, at $x \in C$ and $x^* \in \partial^< f(x_0)$, is defined by

$$\phi(x, x^*) = \sup \big\{ \langle x^*, x - y \rangle \ s.t. \ g_i(y) \leq 0, \ i \in \{1, \ldots, n\} \big\}.$$

The following theorem provides a connection between the value of the gap function at (x_0, x_0^*) and the optimality of x_0.

Theorem 2.4. **(i):** *$\phi(x, x^*) \geq 0$ for each $x \in C$ and $x^* \in \partial^< f(x)$.*

(ii): *If $\phi(x_0, x_0^*) = 0$ for some $x_0 \in C$ and $x_0^* \in \partial^< f(x_0)$, then x_0 is an optimal solution to $(\mathcal{M})$.*

(iii): *Let $x_0 \in C$ be an optimal solution to $(\mathcal{M})$. Under the hypothesis of Theorem 2.1, one has $\phi(x_0, x_0^*) = 0$ for some $x_0^* \in \partial^< f(x_0)$.*

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AN INFEASIBLE INTERIOR POINT ALGORITHM FOR LINEARLY CONSTRAINED CONVEX OPTIMIZATION

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Abstract. In this paper, an infeasible interior point algorithm for solving linearly constrained convex optimization problems is presented. The new algorithm is performed on a wide neighborhood of the central path and searches for the optimizers along the ellipses that approximate the entire of the central path. We compute the polynomial complexity of the proposed algorithm and show that our algorithm has the best known iteration complexity bound.

1. Introduction

The linearly constrained convex optimization (LCCO) problems are one of the fundamental problems in mathematical programming. The standard form of the LCCO problems is given by:

$$(P) \qquad \begin{aligned} &\min f(x) \\ &s.t.: \ Ax = b, \ x \geq 0, \end{aligned}$$

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where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex and continuously differentiable function, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The wolf dual problem of (P) is given by:

$$(D) \qquad \begin{aligned} &\max b^T y + f(x) - x^T \nabla f(x) \\ &s.t.: \ A^T y + s - \nabla f(x) = 0, \ s \geq 0, \end{aligned}$$

where $\nabla f(x)$ denotes the gradient vector of $f(x)$.

In recent years, many researchers have attempted to find some methods which have the best results in solving related mathematical programming problems. The class of Interior Point Methods (IPMs) is one of these kind of methods which have many followers. This class can be divided into two main categories: feasible and infeasible interior point methods. An important work with subject feasible IPMs goes back to work proposed by Bai et al. [1]. They showed that their algorithm for large and small-update methods have $O(\sqrt{n}\log n \log \frac{n}{\varepsilon})$ and $O(\sqrt{n}\log n \log \frac{n}{\varepsilon})$ iteration complexity bounds. One other hand, Roos [5] presented an interesting paper on infeasible area of the IPMs. Also, to obtain efficient algorithms in infeasible area of IPMs, several approach are suggested. For examples Yang [3] introduced the concept of arc search IPMs that they are searching for optimizers along an ellipsoidal approximation of the central-path. To overcome underlying problem of the Yangs algorithms, that is, the choice of an initial feasible solution, Pirhaji [2] showed that an arc search infeasible interior-point algorithm for linear complementarity problems which is well defined and worked with an arbitrary initial point.

In this paper, an arc infeasible interior point algorithm for solving LCCO problems is proposed. To analyze this new algorithm, we apply the Ai-Zhangs neighborhood [4] and use the new idea proposed by Yang [3] to obtain the search direction. Finally, we compute the worst case iteration complexity bound for the new proposed algorithm.

2. Central-path

Throughout this paper, without loss of generality, we have the following assumption:

- **•:** The matrix A is full rank i.e., $\text{rank}(A) = m \leq n$;
- **•:** f is convex and twice continuously differentiable function.

The second assumption implies that the hessian matrix $\nabla^2 f(x)$ of f is positive semidefinite. Assuming an initial point (x^0, y^0, s^0) and considering (x^k, y^k, s^k) as a current iteration of the algorithm, we define the following system:

$$Ax^k - b = r_p^k, \quad A^T y^k - \nabla f(x) + s^k - c = r_d^k, \quad x(\mu)s(\mu) = \mu x^k s^k. \ (2.1)$$

An infeasible central path is defined as:

$$\mathcal{H} := \{(x(\mu), y(\mu), s(\mu)) : \ (x(\mu), y(\mu), s(\mu)) = \vec{a}\cos(\alpha) + \vec{b}\sin(\alpha) + \vec{c}\}$$

where $\vec{a} \in \mathbf{R}^{2n+m}$ and $\vec{b} \in \mathbf{R}^{2n+m}$ are the axes of the ellipse and they are perpendicular to each other, $\vec{c} \in \mathbf{R}^{2n+m}$ is the center of the ellipse. Similar to [4], we define the following ℓ_2-neighborhood of the central path as below:

$$\mathcal{N}(\tau, \beta) := \{(x, s) \in \mathbf{R}^{2n}_{++} : \ ||(\tau\mu e - xs)^+|| \leq \beta\tau\mu\}. \tag{2.2}$$

The next lemma presents our new point.

Lemma 2.1. *Suppose that the new point generated by the above approach; thus this new point has the following formulate:*

$$\begin{aligned} x(\alpha) : &= x - \dot{x}\sin(\alpha) + \ddot{x}(1 - \cos(\alpha)); \\ y(\alpha) : &= y - \dot{y}\sin(\alpha) + \ddot{y}(1 - \cos(\alpha)); \\ s(\alpha) : &= x - \dot{x}\sin(\alpha) + \ddot{s}(1 - \cos(\alpha)), \end{aligned}$$

where $(\dot{x}, \dot{y}, \dot{s})$ *and* $(\ddot{x}, \ddot{y}, \ddot{s})$ *are computed via the following procedure.*

$$(\dot{x}, \dot{y}, \dot{s}) = \lambda(\dot{x}_1, \dot{y}_1, \dot{s}_1) + (1 - \lambda)(\dot{x}_2, \dot{y}_2, \dot{s}_2), \tag{2.3}$$

where $\lambda := \max\{\lambda \in [0, 1] : \dot{x}^T\dot{s} \geq -\frac{1}{4}(1 + \beta\tau)n\mu\}$ *and* $(\dot{x}_1, \dot{y}_1, \dot{s}_1)$ *and* $(\dot{x}_2, \dot{y}_2, \dot{s}_2)$ *are obtained from:*

$$\begin{aligned} &A\dot{x}_1 = r_p, \quad -H\dot{x}_1 + A^T\dot{y}_1 + I\dot{s}_1 = r_d, \quad S\dot{x}_1 + X\dot{s}_1 = r_c; \\ &A\dot{x}_2 = 0, \quad -H\dot{x}_2 + A^T\dot{y}_2 + I\dot{s}_2 = 0, \quad S\dot{x}_2 + X\dot{s}_2 = r_c, \end{aligned}$$

where the vector r_c *is defined as:* $r_c := -\left[(\tau\mu e - sx)^- + \sqrt{}(\tau\mu e - xs)^+\right]$. *Moreover,* $(\ddot{x}, \ddot{y}, \ddot{s})$ *is obtained from the following system:*

$$A\ddot{x} = 0, \quad -H\ddot{x} + A^T\ddot{y} + \ddot{s} = 0, \quad s\ddot{x} + x\ddot{s} = -2\dot{x}\dot{s}. \tag{2.4}$$

3. Algorithm

In this section, we present an arc infeasible IPM algorithm for LCCO problems.

Algorithm 1. Infeasible-IPM with arc-search for LCCO problems

Step 0

Fix some parameters $\theta \in (0, \frac{1}{2+\sqrt{2}}]$, $\epsilon > 0$, $\tau \in (0, \frac{1}{4}]$

initial point $(x^0, y^0, s^0) \in \mathcal{N}(\tau, \beta)$

Step 1 If

$$||r_p^k|| \leq \varepsilon, \quad ||r_d^k|| \leq \varepsilon \quad (x, s) > 0,$$

stop. Otherwise go to Step 2.

Step 2 By using (2.3) and (2.4), compute $(\dot{x}, \dot{y}, \dot{s})$ and $(\ddot{x}, \ddot{y}, \ddot{s})$.
Step 3 compute the largest positive $\hat{\alpha_k} \in (0, \frac{\pi}{2}]$ such that for all $\alpha \in (0, \hat{\alpha_k}]$, the following inequality are true:

$$\mu(\hat{\alpha_k}) \leq (1 - \frac{\sin(\alpha)}{2})\mu_k \tag{3.1}$$

$$x(\alpha)^T s(\alpha) \geq (1 - \sin(\alpha))x_k^T s_k \tag{3.2}$$

$$(x(\alpha), y(\alpha), s(\alpha)) \in \mathcal{N}(\tau, \beta) \tag{3.3}$$

Step 4 Let the new iterate $(x^{k+1}, y^{k+1}, s^{k+1}) := (x(\hat{\alpha_k}), y(\hat{\alpha_k}), s(\hat{\alpha_k}))$ and compute $\mu_{k+1} = \frac{x_{k+1}^T s_{k+1}}{n}$. Set $k := k + 1$ and go to Step 1.
end

To obtain the complexity result of the algorithm, we first give two technical lemma presented in [2].

Lemma 3.1. *Suppose that* $\sin(\hat{\alpha})$ *is the largest positive such that (3.1) holds. Then, we have:* $\sin(\hat{\alpha}) \geq \sin(\hat{\alpha_0}) := \frac{\beta\tau}{\sqrt{n}\sqrt{\omega}}$.

Lemma 3.2. *Suppose that* $\sin(\hat{\alpha_0}) = \frac{\beta\tau}{\sqrt{n}\sqrt{\omega}}$. *Then for all* α *such that* $\sin(\alpha) \in [0, \sin(\hat{\alpha_0})]$, *we conclude that:* $(x(\alpha), y(\alpha), s(\alpha)) \in \mathcal{N}(\tau, \beta)$.

Theorem 3.3. *Suppose that* $\sin(\hat{\alpha_0}) = \frac{\beta\tau}{\sqrt{n}\sqrt{\omega}}$ *and* $(x(\alpha), y(\alpha), s(\alpha)) \in \mathcal{N}(\tau, \beta)$. *Then, the total number of iterations to get an* ε*-solution of problems (P) and (D), i.e., a solution that satisfies* $x^T s = n\mu \leq \varepsilon$, *is bounded by* $O(n \log \varepsilon^{-1})$.

The numerical result and simulation the performing algorithm will be presented in speech.

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A NUMERICAL METHOD FOR SOLVING A CLASS OF VARIATIONAL PROBLEMS

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Abstract. Variational problems can have piecewise smooth optimal solutions. This paper presents a numerical direct method, based on the extension of the well-known Ritz method, for solving variational problems with piecewise smooth solution.

1. Introduction

It is known that, the variational problem

$$min \quad J[u] = \int_{t_0}^{t_1} F(t, u, u')dt, \quad u(t_0) = u_0, \quad u(t_1) = u_1, \tag{1.1}$$

can have optimal solution of the following form in the space of piecewise smooth functions.

$$u^*(t) = \begin{cases} u_1^*(t), & t_0 \leq t \leq p^*, \\ u_2^*(t), & p^* \leq t \leq t_1, \end{cases} \tag{1.2}$$

where u_1^* and u_2^* are smooth functions, $u_1^*(p^*) = u_2^*(p^*)$ and the function F is supposed to be continuously differentiable with respect to its arguments. The functional J is supposed to have a minimizing solution

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of the form (1.2), on $E[t_0, t_1]$, where

$$E[t_0, t_1] = \{u(t) : u(t) \text{ is piecewise smooth}, \ u(t_0) = u_0, \ u(t_1) = u_1\}. \tag{1.3}$$

In this paper, by modifying the Ritz direct method [1, 2, 3, 4], we present a method for finding approximate solutions of the form (1.2) for (1.1). Our method is inspired by the extended Ritz method [5, 6, 7]. As a direct method based on the discritization techniques, the interested reader can refer to [8].

2. Extension of the Ritz method

Consider expansions

$$u_{1,k}(t) := {C_{1,k}}^T.\Psi_{1,k}(t) + u_0$$

and

$$u_{2,k}(t) := {C_{2,k}}^T.\Psi_{2,k}(t) + u_1,$$

$$\Psi_{1,k}(t) := (\phi_j t)_{0\le j\le k}, \quad \Psi_{2,k}(t) := (\phi_j(1-t))_{0\le j\le k}, \tag{2.1}$$

$$C_{1,k} := (c_{1,j})_{0\le j\le k}, \quad C_{2,k} := (c_{2,j})_{0\le j\le k},$$

$$u_k(t) := \begin{cases} u_{1,k}(t), & 0 \le t \le p_k, \\ u_{2,k}(t), & p_k \le t \le 1. \end{cases} \tag{2.2}$$

Here, ϕ_js, $j \in \{0\} \bigcup \mathbb{N}$ are polynomial basis functions and $c_{i,j}s$, $i = 1, 2$, $j = 0, 1, \ldots, k$ and p_k are real unknowns. By substituting u_k in (1.1), we achieve

$$J[p_k, C_{1,k}, C_{2,k}] = \int_0^{p_k} F(t, u_{1,k}, u'_{1,k})dt + \int_{p_k}^1 F(t, u_{2,k}, u'_{2,k})dt. \tag{2.3}$$

If $c_{i,j}$s and p_k are determined by minimizing the function J with respect to the constraint $u_{1,k}(p_k) = u_{2,k}(p_k)$, then we achieve the function u_k, which approximates minimum value of J in (2.3) and also is in the form of (1.2). The following theorem demonstrates the convergence of the method.

Theorem 2.1. *Let $\mu^* = J[u^*]$ and also let $\mu_k = J[u_k]$, then*

$$\lim_{k\to\infty} \mu_k = \mu^*.$$

3. Quadratic variational problems

In this section, we focus on the following quadratic class of variational problem (1.1)

$$J[u] = \frac{1}{2}\int_0^1 B(U,U)dt - \int_0^1 L(U)dt + C, \tag{3.1}$$

where C is a real constant, $U := (u, u')$, and the mappings $L : X \to L^2[0,1]$ and $B : X \times X \to L^2[0,1]$ are considered to be linear and bilinear respectively. Here, $X := L^2[0,1] \times L^2[0,1]$ is equipped with the following product norm $\|(f_1, f_2)\| := (\|f_1\|_{L^2}{}^2 + \|f_2\|_{L^2}{}^2)^{\frac{1}{2}}$.

We also define mappings $\tilde{B}$ and $\tilde{L}$ as follows

$$\tilde{B} : X \times X \to \mathbb{R}, \quad (U,V) \mapsto \int_0^1 B(U,V)dt,$$

$$\tilde{L} : X \to \mathbb{R}, \quad U \mapsto \int_0^1 L(U)dt.$$

The functional $\tilde{B}$ is supposed to have the following properties:

(i) *Boundedness.* There exists a constant $d > 0$ such that $\mid \tilde{B}(U,V) \mid \leq d \parallel U \parallel \parallel V \parallel$.

(ii) *Strong positivity.* There exists $c > 0$ such that $c \parallel U \parallel^2 \leq \tilde{B}(U,U)$.

The mapping $\tilde{L}$ is supposed to be bounded.

Theorem 3.1. *A necessary and sufficient condition for u^* to be a unique minimizer of the functional (3.1) over $E[0,1]$ is to satisfy the following variational equality*

$$\tilde{B}(U^*, V) = \tilde{L}(V), \quad v \in E^*[0,1], \tag{3.2}$$

where $U^* = (u^*, u^{*\prime})$, $V = (v, v')$,

$$E^*[0,1] := \{u(t) : u(t) \text{ is piecewise smooth, } u(0) = u(1) = 0\}. \tag{3.3}$$

Now we present an approximate method for solving the variational equality (3.2). Let $U_k := (u_k, u_k{}')$, where u_k is defined in (2.2). Also let

$$\Phi_{1,j}(t) := \begin{cases} \alpha_j(t), & 0 \leq t \leq p_k, \\ 0, & p_k < t \leq 1, \end{cases} \quad \Phi_{2,j}(t) := \begin{cases} 0, & 0 \leq t < p_k, \\ \beta_j(t), & p_k \leq t \leq 1, \end{cases} \tag{3.4}$$

where $\alpha_j(t) := t\phi_j(t)$ and $\beta_j(t) := (1-t)\phi_j(t)$. Then

$$\tilde{B}(U_k, \mu_{i,j}) = \tilde{L}(\mu_{i,j}), \quad 0 \leq j \leq k, \quad i = 1,2, \tag{3.5}$$

$$\mu_{i,j} := (\Phi_{i,j}, \Phi_{i,j}{}'),$$

forms a system of $2k+2$ equations and $2k+3$ unknowns. Note that, p_k is an unknown which should be determined. We rewrite system (3.5) in the form (3.6)

$$\int_0^{p_k} B(U_k, \mu_{i,j})dt + \int_{p_k}^1 B(U_k, \mu_{i,j})dt = \int_0^{p_k} L(\mu_{i,j})dt + \int_{p_k}^1 L(\mu_{i,j})dt, \tag{3.6}$$

$$u_{1,k}(p_k) = u_{2,k}(p_k). \tag{3.7}$$

By solving the system (3.6), (3.7), we achieve coefficients of expansion (2.2) and p_k. Now u_k is an approximate minimizer for the functional (3.1).

In Theorem 3.2, we show that with increase in values of k, the approximate solution u_k tends to the exact solution u^*.

Theorem 3.2. *Suppose u_k is derived by solving the system (3.6), (3.7), then*

$$\lim_{k\to\infty} \| u_k - u^* \|_{L^2} = \lim_{k\to\infty} \| u_k' - u^{*\prime} \|_{L^2} = 0.$$

4. Conclusions

A direct method to find approximate solutions for variational problems with piecewise smooth solutions is developed by extending the well-known Ritz method. Although the discussed problem has optimal solution containing a single corner point, the method can be naturally generalized to the cases with multiple corner points.

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EFFICIENT NEURODYNAMIC MODEL FOR FUZZY LINEAR REGRESSION

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Abstract. Fuzzy linear regression models are used to obtain an appropriate linear relation between a dependent variable and several independent variables in a fuzzy environment. In this paper, a neural network model is constructed on the basis of the duality theory to find the approximate parameters. A numerical result shows the performance of the method.

1. Introduction

Regression analysis is one of the most popular methods of estimation. It is applied to evaluate the functional relationship between the dependent and independent variables. Fuzzy regression analysis is an extension of the classical regression analysis in which some elements of the model are represented by fuzzy numbers [2]. In this paper, we present a neural network [4] for solving the fuzzy linear regression model. Finally, we estimate the parameters of a fuzzy regression model and compare our estimation with those previously presented.

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2. Preliminaries

Definition 2.1. A fuzzy number is a fuzzy set $u : \mathbb{R}^1 \to [0,1]$ such that
i. u is upper semi-continuous;
ii. $u(x) = 0$ outside some interval $[a,d]$;
iii.There are real numbers b and c, $a \leq b \leq c \leq d$, for which
1. $u(x)$ is monotonically increasing on $[a,b]$,
2. $u(x)$ is monotonically decreasing on $[c,d]$,
3. $u(x) = 1$, $b \leq x \leq c$.

For $0 < h \leq 1$, a h-level set of a fuzzy number X is defined as $[X]_h = \{x \in \mathbb{R} | \mu_X(x) \geq h\}$, and $[X]_0 = \overline{\bigcup_{h\in(0,1]}[X]_h}$. We denote $[X]_h$ by $[X]_h = \left[[X]_h^L, [X]_h^U\right]$.

A popular fuzzy number is the triangular fuzzy number $u = (u_l, u_m, u_r)$ where u_m denotes the modal value and the real values u_l and u_r represent the left and right points of the triangular fuzzy number.

Definition 2.2. A metric d^* is defined as $d^*(X,Y) = sup_{h>0} d_H([X]_h, [Y]_h)$, where the Hausdorff metric d_H is given by

$$d_H(A,B) = max(sup_{a\in A} inf_{b\in B}||a-b||, sup_{b\in B} inf_{a\in A}||a-b||). \quad (2.1)$$

For two triangular fuzzy numbers $u_1 = (u_{1m}, u_{1l}, u_{1r})$ and $u_2 = (u_{2m}, u_{2l}, u_{2r})$, (2.1) reduces to $d(u_1,u_2)$, where [1]

$$d^2(u_1,u_2) = (u_{1l}-u_{2l})^2 + (u_{1m}-u_{2m})^2 + (u_{1r}-u_{2r})^2. \quad (2.2)$$

3. The fuzzy polynomial regression model

We have postulated that the dependent fuzzy variable Y, is a function of the independent real variables $x_1, x_2, ..., x_n$. More formally

$$f : \mathbb{R}^n \to E,$$

$$Y_i = f(x_{i1}, x_{i2}, ..., x_{in}),$$

where i indexes the observations. The objective is to estimate a fuzzy polynomial regression (FPR) model, express as follows:

$$\overline{Y}_i = A_{l0} + \sum_{j=1}^{n} A_{lj}x_{ij} + \sum_{j=1}^{n}\sum_{k=1}^{n} A_{ljk}x_{ij}x_{ik} + ..., \quad i = 1,2,...,m. \quad (3.1)$$

This full form of mathematical description in Eq.(3.1) can be represented by a fuzzy polynomial consisting of only one variable in the form of

$$\overline{Y}_i = A_0 + A_1x_i + A_2x_i^2 + ... + A_nx_i^n, \quad (3.2)$$

when $\overline{Y}_i = A_0 + A_1 x_i$. We might do it by eye-fitting the line that looks best to us. Unfortunately, different people will draw different polynomials and it would be nice to have a formal method for finding the polynomial that would consistently provide us with the best polynomial possible. What would a best possible polynomial look like? Intuitively, it would seem to have to be a polynomial that fit the data well. That is, the distance of the polynomial from the observations should be as small as possible. Let $A_0, A_1, ..., A_n$ denote the list of regression coefficients (parameters). A_0 is an optional intercept parameter and $A_l, ..., A_n$ are weights or regression coefficients corresponding to x_i. Then fuzzy polynomial regression is given by

$$\overline{Y}_i = A_0 + A_1 x_i + A_2 x_i^2 + ... + A_n x_i^n, \tag{3.3}$$

where i indexes the different observations and $A_0, A_1, ..., A_n$ are fuzzy numbers. We are interested in finding $A_0, A_1, ..., A_n$ of fuzzy polynomial regression such that $\overline{Y}_i$ approximates for all $i = 1, 2, ..., m$, closely enough according to some norm $\|.\|$, i.e.,

$$min\left\|[\overline{Y}_i]_h^\dagger - [Y_i]_h^\dagger\right\|, \quad h \in [0,1], \quad for\ all\ i = 1, 2, ..., m, \tag{3.4}$$

where $\dagger$ means we have this equation for U (upper limit) and L (lower limit) together, independently.

4. A neural network model

From previous section, we see that the fuzzy polynomial regression model can be reduced to the following optimization problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad g(x) \leq 0. \end{aligned} \tag{4.1}$$

We propose the following recurrent neural network for solving (4.1) as

$$\begin{aligned} \frac{dx}{dt} &= -(\nabla f(x) + \nabla g(x)^T (u + g(x))^+), \\ \frac{du}{dt} &= (u + g(x))^+ - u, \end{aligned} \tag{4.2}$$

Theorem 4.1. [4] *The proposed neural network model in (4.2) is globally stable in the Lyapunov sense and is globally convergent.*

5. A numerical example

Example 5.1. We now apply the proposed method to fit the fuzzy linear regression model to the data taken from Hong and Hwang [3], as shown in Table (1). By using (4.2), the fuzzy linear regression model obtained by our method is as follows:

$$y_i = (5.2491, 4.8948, 4.5406) + (1.1383, 1.1954, 1.2524)x_i$$

TABLE 1. The data and error in estimation for Example (5.1)

Obs.	Res.	Pre.	Hong-Hwang	Kao-Chyu	NN
1	(-2.1, -1.6, -1.1)	1.0	2.0160	2.8875	0.9113
2	(-2.3, -1.8, -1.3)	3.0	2.0480	1.9324	0.7473
3	(-1.5, -1.0, -0.5)	4.0	3.1537	1.9943	1.0471
4	(0.7, 1.2, 1.7)	5.6	4.6965	1.9458	0.8928
5	(1.2, 2.2, 3.2)	7.8	3.4238	2.6143	1.7993
6	(5.8, 6.8, 7.8)	10.2	4.1836	1.4666	0.8685
7	(9.0, 10.0, 11.0)	11.0	4.8699	3.0077	1.9537
8	(9.0, 10.0, 11.0)	11.5	4.9699	2.4298	1.6401
9	(9.0, 10.0, 11.0)	12.7	5.4516	1.4254	0.5342
Total error			34.8130	19.7039	10.3944

The errors of the observations are shown in Table (5.1). We see that the sum of errors in our method is considerably less than the other methods.

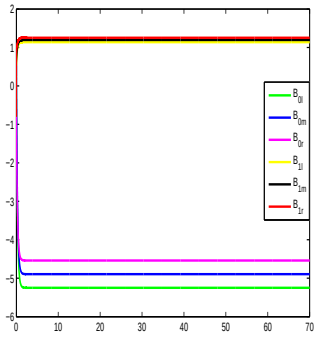

FIGURE 1. Transient behaviors of the neural network (4.2) with a random initial point.

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A MEASURE THEORETICAL METHOD FOR DESIGNING GENERAL 3-D OPTIMAL SHAPE IN SPHERICAL COORDINATES

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Abstract. In this paper, the goal is to determine a bounded shape located over the (x, y)-plane, such that its projection in the (x, y)-plane and its volume are given and it minimizes some given surface integral in spherical coordinates.To solve such a problem, we somehow extend the embedding process in the space of Radon measures. First, the problem is converted into an infinite dimensional linear program. Then this problem is reduced to a finite dimensional nonlinear program using approximation scheme. Finally, the solution of this new problem is used to construct a nearly optimal smooth surface by applying a kind of outlier detection method and MATLAB smooth curve fitting toolbox. In comparison to the other methods, this approach has some advantages; a numerical example is given to illustrate the proposed method and its advantages.

2010 *Mathematics Subject Classification.* Primary 93B51; Secondary 93C10, 90C30.

Key words and phrases. Optimal shape design, Radon measure, Nonlinear programming, Spherical coordinates, PSO algorithm.

* Speaker.

1. Introduction

In two recent decades, some authors have considered a general and linear method based on an embedding process for solving OSD problems in two dimensions [2, 3]. But a huge number of problems are based on three-dimensional; also, we know that spherical coordinates has many remarkable advantages. Regarding these facts, in [1], for the first time, an extended version of the shape-measure method in spherical coordinates was presented. Now here, a modification on this method will be extended for determining general 3-D optimal shape. We suppose that the boundary of the general shape C is consistant of two parts; the first is an unknown smooth surface S defined by $\phi = f(\rho, \theta)$ and the second is the given region D, the projection of S in (x, y)-plane. Additionally D is surrounded by a simple and closed curve ∂D, demonstrated by $r = h(\theta)$, $0 \leq \theta \leq 2\pi$ and passes through a given point $B = (h(\theta_0), \frac{\pi}{2}, \theta_0)$.

2. Statement of the problem

We try to determine an admissible surface S which minimizes $I(S) = \int_S f_0(\rho, \phi, \theta) d\sigma$, where f_0 is a given continuous (measurable) function defined on S. To solve this problem, by applying embedding process, we transform it to a classical optimal control problem; we consider $\phi(\rho, \theta)$ as a trajectory function and choosing artificial control functions as $u_1 \equiv u_1(\rho, \theta) = f_\rho$, $u_2 \equiv u_2(\rho, \theta) = f_\theta$, $u_3 \equiv u_3(\rho, \theta) = f_{\rho\theta}$. Thus, we can convert the mentioned OSD problem as follows:

$$
\begin{aligned}
\underset{P\in W}{Min} \quad & I(P) = \iint_D f_1(\rho, \phi, \theta, u_1, u_2, u_3)\rho d\rho d\theta \\
S.to: \quad & \iint_D \Phi^g \rho d\rho d\theta = \rho_0 \Delta_\Phi, \quad \forall \Phi \in C'(B); \\
& \iint_D \Psi \rho d\rho d\theta = 0 \ , \qquad \forall \psi \in \wp(D^\circ); \\
& \iint_D g\rho d\rho d\theta = a_g \ , \forall g \in C_1(\Omega); \\
& \iint_D \frac{(\rho u_1 - 1 - \theta u_2 \cot\phi)\rho \sin\phi}{u_2} \rho d\rho d\theta = 3V_0; \\
& \iint_D (\frac{\partial K}{\partial \rho} - \frac{\partial K}{\partial \phi} u_1)\rho d\rho d\theta = 0 \ , \qquad \forall\, K(\phi, \theta) \in C^1(D \times A); \\
& \iint_D (\frac{\partial L}{\partial \rho} - \frac{\partial L}{\partial \phi} u_2)\rho d\rho d\theta = 0 \ , \quad \forall\, L(\rho, \phi) \in C^1(D \times A);
\end{aligned}
\tag{2.1}
$$

Here, the first, second and third constraints are similar to [4]. The fourth constraint indicates volume of C and the two last constraints are given to show the dependency of controls. Now, we can express (2.1) in measure space by defining the following positive linear functional:

$$\Lambda_P : F \in C(\Omega) \to \iint_D F(\rho, \phi, \theta, u_1, u_2, u_3)\rho d\rho d\theta.$$

Then we follow Rubio in [4] to approximate the solution of (2.1) with a following nonlinear programming problem by using Riesz representation theorem, atomic measures and approximation scheme:

$$\begin{aligned}
Min \ & \sum_{j=1}^{M} \alpha_j f_1(q_j^*) \\
S.to: & \sum_{j=1}^{M} \alpha_j \Phi_i^g(q_j^*) = \rho_0 \Delta_{\Phi_i}, \quad i = 1,\ 2,\ ...,\ M_1; \\
& \sum_{j=1}^{M} \alpha_j \Psi_h(q_j^*) = 0\ , \qquad h = 1, 2, ..., M_2\ ; \\
& \sum_{j=1}^{M} \alpha_j g_{sk}(q_j^*) = a_{g_{sk}}\ , \quad s = 1, 2, ..., R_1, k = 1, 2, ..., R_2\ ; \qquad (2.2) \\
& \sum_{j=1}^{M} \alpha_j \frac{(\rho u_1 - 1 - \theta u_2 \cot\phi)\rho \sin\phi}{u_2}(q_j^*) = 3V_0\ ; \\
& \sum_{j=1}^{M} \alpha_j H_{1m}(q_j^*) = 0\ , \qquad m = 1, 2, ..., M_4\ ; \\
& \sum_{j=1}^{M} \alpha_j H_{2l}(q_j^*) = 0\ , \qquad l = 1, 2, ..., M_5\ ; \\
& \alpha_j \geq 0\ ,\ j = 1,\ 2,\ ...,\ M,
\end{aligned}$$

where $M = M_1 + M_2 + (R_1 \times R_2) + M_4 + M_5 + 1$.

3. Optimization algorithm

To have a good approximation, it is necessary that $M_1, M_2R_1, R_2, M_4, M_5$ are selected large enough; thus, the nonlinear problem (2.2) is a large scale problem and metaheuristic algorithms can be suitable enough to solve it. Among them, we apply PSO; then, the procedure to construct a piecewise constant optimal control and trajectory functions from the

solution of (2.2), can be done based on the analysis in [4].

Example: We incline to determine an unknown symmetric three-dimensional bounded shape with minimum surface area that is with volume $\frac{32\pi}{3}$ where D, is a circle by radius 2. Provided that there is also a given point on the boundary of the region D as $(2, \frac{\pi}{2}, 0) = (2, \frac{\pi}{2}, 2\pi)$. We choose $U_1 = U_2 = U_3 = [-5, 5]$, $M_1 = 4$, $M_2 = 9$, $R_1 = 3$, $R_2 = 6$ and $M_4 = M_5 = 1$. After solving the related NLP by PSO, we obtain the optimal points. Then, we reject the outliers with appropriate method and fit a surface on remainder points (Fig.1).

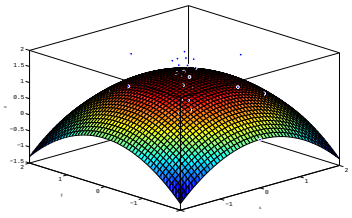

FIGURE 1. The nearly optimal surface

4. Conclusion

We have extended an extended embedding method (embedding in Radon measures spaces) for spherical coordinates to be able to solve 3-D shape optimization problems. In this new method, the nearly optimal shape can be determined, with one step less in approximation, from the solution of the corresponding simple finite dimensional nonlinear programming problem, instead of finite linear programming. By this method, a smoother shape can be determined, since it is able to employ a suitable algorithm for rejecting the outlier data.

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BACKSTEPPING DESIGN FOR A CLASS COUPLED PARABOLIC PDES WITH NON-CONSTANT DIFFUSIONS

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Abstract. This paper aims to plan a backstepping feedback controller for a class of coupled parabolic partial differential equations with different non-constant diffusions. We design a backstepping controller which guarantees the stability of the system. Also, the numerical simulation are presented.

1. Introduction

There are numerous industrials systems demonstrated by coupled parabolic partial differential equations (PDEs) systems. In [1] the authors utilized backstepping approach to stabilize a set of coupled PDEs with constant diffusion coefficients. A new way based on backstepping design is presented by [2]. In [3] the authors represented a backestepping method for one dimensional parabolic cases with space dependent diffusivity and time varying condition. In this paper, we design a backstepping feedback controller for stabilization of coupled parabolic PDEs with nonconstant diffusion coefficients by utilizing boundary control.

2010 *Mathematics Subject Classification.* Primary 47A55; Secondary 39B52, 34K20, 39B82.

Key words and phrases. Backstepping control, Lyapunov function, Coupled parabolic partial differential equations.

* Speaker.

2. Problem statement

Consider a set of systems by n coupled parabolic PDEs with non-constant diffusions with Neumann kind boundary conditions:

$$U_t(x,t) = \Theta(x)U_{xx}(x,t) + \Psi U(x,t), \tag{2.1}$$
$$U_x(x,t) = 0, \tag{2.2}$$
$$U(1,t) = U_c(t). \tag{2.3}$$

The state vector is, $U(x,t) = (u_1(x,t), u_2(x,t), \dots, u_n(x,t)) \in (L^2(0,1))^n$. The input vector is $U_c(1,t) = (u_{c_1}(1,t), u_{c_2}(1,t), \dots, u_{c_n}(1,t)) \in (L^2(0,1))^n$. $\Theta(x)$ is a positive diagonal $n \times n$ matrix, whose components $\theta_i(x)$ for $i \in N_n$, represent non-constant diffusions of the system, and Ψ is diagonal $n \times n$ matrix, whose components ψ_i for $i \in N_n$, display the reaction term, and $n \in N$ is the number of coupled PDEs.

3. Backstepping controller design

The systems (2.1)-(2.3) is unstable. For designing stable controller, we introduce the following backstepping transformation.

$$W(x,t) = U(x,t) - \int_0^x K(x,y)U(y,t)dy, \tag{3.1}$$

We apply (3.1) to transfer (2.1)-(2.3) into following stable target sytem:

$$W_t(x,t) = \Theta(x)W_{xx}(x,t) - CW(x,t), \tag{3.2}$$
$$W_x(x,t) = 0, \tag{3.3}$$
$$W(1,t) = 0, \tag{3.4}$$

where C is diagonal matrix parameter with components c_i, $i \in N_n$. The boundary condition (3.1) yields the feedback controller in the form:

$$U(1,t) = \int_0^1 K(1,y)U(y,t)dy. \tag{3.5}$$

It is assumed that $K(x,y) = k_i(x,y)I_{n\times n}$. The following lemma establishes a condition that guarantees stability of the system (3.2)-(3.4).

Lemma 3.1. *([3]). System (3.2)-(3.4) is exponentially stable with the following condition*

$$c_i \geq \frac{\theta''_{i\max}}{2}, \quad \textit{where} \quad \theta''_{i\max} = \max \theta''_i(x), \quad x \in [0,1], \ i \in N_n. \tag{3.6}$$

By applying (3.1) and replacing it in (2.1)-(2.3) and (3.2)-(3.4) we have:

$$\theta_i(x)k_{ixx}(x,y) - (\theta_i(y)k_i(x,y))_{yy} = (\psi_i + c_i)k_i(x,y), \ i \in N_n, \tag{3.7}$$

and
$$k_{iy}(x,0) = -\frac{\theta_i'(0)}{\theta_i(0)}k_i(x,0),\ i \in N_n, \tag{3.8}$$
$$k_i(x,x) = -\frac{1}{2\sqrt{\theta_i(x)}}\int_0^x \frac{\psi_i + c_i}{\sqrt{\theta_i(\tau)}}d\tau. \tag{3.9}$$

By considering
$$\hat{k}_i(\overline{x_i},\overline{y_i}) = \theta_i(y)k_i(x,y), \quad \overline{x_i} = \phi_i(x),\ \overline{y_i} = \phi_i(y),$$
$$\phi_i(x) = \sqrt{\theta_i(0)}\int_0^x \frac{1}{\sqrt{\theta_i(\tau)}}d\tau, \tag{3.10}$$
$$\overline{k_i}(\overline{x_i},\overline{y_i}) = (\theta_i(x)\theta_i(y))^{\frac{-1}{4}}\hat{k}_i(\overline{x_i},\overline{y_i}), \tag{3.11}$$

the PDE (3.7)-(3.9), becomes:
$$\theta_i(0)(\overline{k}_{i\bar{x}_i\bar{x}_i}(\overline{x_i},\overline{y_i}) - \overline{k}_{i\bar{y}_i\bar{y}_i}(\overline{x_i},\overline{y_i})) =$$
$$(\frac{3}{16}(\frac{\theta_i'(x)^2}{\theta_i(x)} - \frac{\theta_i'(y)^2}{\theta_i(y)}) + \frac{1}{4}(\theta_i''(y) - \theta_i''(x)) + (\psi_i + c_i))\overline{k_i}(\overline{x_i},\overline{y_i}), \tag{3.12}$$
$$\overline{k_i}(\overline{x_i},0) = 0, \tag{3.13}$$
$$\overline{k_i}(\overline{x_i},\overline{x_i}) = -\frac{(\psi_i + c_i)}{2\sqrt{\theta_i(0)}}\overline{x_i}. \tag{3.14}$$

By solving ODEs, $\frac{3}{16}(\frac{\theta_i'(x)^2}{\theta_i(x)} - \frac{1}{4}\theta_i''(x)) = c1_i$, Eq. (3.12) can be derived:
$$\theta_i(0)(\overline{k}_{i\bar{x}_i\bar{x}_i}(\overline{x_i},\overline{y_i}) - \overline{k}_{i\bar{y}_i\bar{y}_i}(\overline{x_i},\overline{y_i})) = ((\psi_i + c_i))\overline{k_i}(\overline{x_i},\overline{y_i}),$$
and $\theta_i(x) = \varepsilon_{0i}(1 + \varepsilon_{1i}(x - x_{0i})^2)^2$, where $\varepsilon_{0i}, \varepsilon_{1i}, x_{0i}$, are arbitrary constants, and $c1_i = -\varepsilon_{0i}\varepsilon_{1i}$. The PDEs (3.12)-(3.14) can be solved (for detail, please see [2]):
$$\overline{k_i}(\overline{x_i},\overline{y_i}) = -\overline{y_i}\frac{(\psi_i + c_i)}{\sqrt{\theta_i(0)}}\frac{I_1(\sqrt{\frac{\psi_i + c_i}{\theta_i(0)}(\overline{x_i}^2 - \overline{y_i}^2)})}{\sqrt{\frac{\psi_i + c_i}{\theta_i(0)}(\overline{x_i}^2 - \overline{y_i}^2)}}, \tag{3.15}$$

where I_1 is a modified Bessel function of order one. By transformation $\overline{k_i}(\overline{x_i},\overline{y_i}) = (\theta_i(x)\theta_i(y))^{\frac{-1}{4}}\theta_i(y)k_i(x,y)$, Eq. (3.15) can be derived:
$$k_i(x,y) = -\frac{\theta_i(x)^{\frac{1}{4}}}{\theta_i(y)^{\frac{3}{4}}}\frac{(\psi_i + c_i)}{\sqrt{\theta_i(0)}}\overline{y_i}\frac{I_1(\sqrt{\frac{\psi_i + c_i}{\theta_i(0)}(\overline{x_i}^2 - \overline{y_i}^2)})}{\sqrt{\frac{\psi_i + c_i}{\theta_i(0)}(\overline{x_i}^2 - {}_i\overline{y}^2)}}, \tag{3.16}$$
$$\overline{x_i} = \phi_i(x), \quad \overline{y_i} = \phi_i(y), \tag{3.17}$$
$$\phi_i(\xi) = \frac{1 + \varepsilon_{1i}x_{0i}^2}{\sqrt{\varepsilon_{1i}}}(arctan(\sqrt{\varepsilon_{1i}}(\xi - x_{0i})) + arctan(\sqrt{\varepsilon_{1i}x_{0i}})). \tag{3.18}$$

Example 3.2. Consider the following parabolic PDEs systems:

$$u_{1t}(x,t) = \frac{1}{4}((1+(x-\frac{1}{2})^2))^2 u_{1xx}(x,t) + 0.85u_1(x,t), \quad (3.19)$$

$$u_{2t}(x,t) = \frac{1}{2}((1+(x-\frac{1}{4})^2))^2 u_{2xx}(x,t) + 1.705u_2(x,t), \quad (3.20)$$

$$u_{1x}(0,t) = 0, \quad (3.21)$$

$$u_{2x}(0,t) = 0, \quad (3.22)$$

$$u_1(1,t) = 5sin(t), \quad (3.23)$$

$$u_2(1,t) = 10sin(2t), \quad (3.24)$$

$$u_1(x,0) = u_2(x,0) = sin(\pi x) + sin(3\pi x). \quad (3.25)$$

The control of the system (3.19)-(3.25) will be designed by applying approach in Section (3) and lemma (3.1). For simulating, the Matlab software is applied.

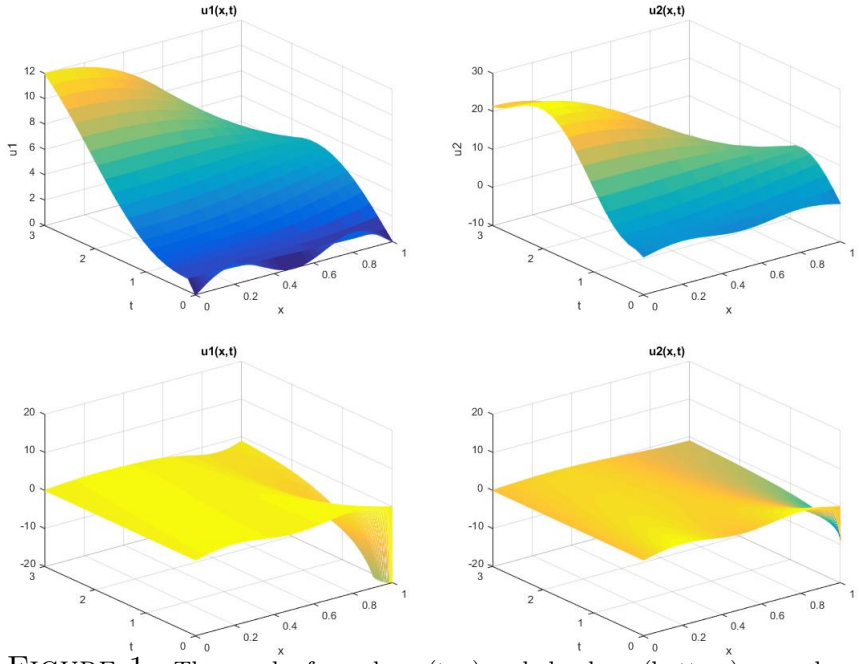

FIGURE 1. The graph of openloop (top) and closeloop (bottom) u_1 and u_2.

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THE PERFORMANC OF ASYMMETRIC LEAST SQUARES SUPPORT VECTOR MACHINES FOR SOLVING INITIAL VALUE PROBLEMS

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Abstract. In this paper, we introduce a new method for solving initial value problems and show how to obtained approximate solution for initial value problem is found by ALSSVM prescription. This process is suitable for solving differential equations with initial value conditions. Some numerical simulations are illustrated to show the proposed method.

1. Introduction

The ordinary differential equations (ODEs) plays an important role in engineering, medical sciences and medicine. Often, ODEs construct to modeling of phisycal phenomenas. There are many algorithms for solving ODEs([2]-[4]). The Support Vector Machines are set of learning machine methods used for solving problems in Classification, Regression and Detection consists of classifier algorithms based on Statistical

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Key words and phrases. Support vector machine; Initial Value Problem; Asymmetric Least Squares Support Vector Machines.

* Speaker.

Learning Theory. Also, ALSSVM algorithm is an excellent tool for binary data classification[1]. In this paper, we ALSSVM algorithm for earn approximation solution of ODEs.

2. ALSSVM method

Consider a data set $z = \{(x_i, y_i)\}_{i=1}^m$, where $x_i \in \mathbb{R}^d$ and $y_i \in \{-1, +1\}$. Now, we consider the following pinball loss:

$$L_p^{pin}(t) = \begin{cases} pt & \text{if} \quad t \geq 0 \\ -(1-p)t & \text{if} \quad t < 0 \end{cases}$$

Which is related to the p lower value and $0 \leq p \leq 1$ [1]. We can maximize the quantile distance by following pinball loss SVM proposed by Huang et al. [4]:

$$\min \frac{1}{2}\|w\|^2 + \frac{C}{2}\sum_{i=1}^{m} L_p^{pin}(1 - y_i(w.\phi(x_i) + b))$$

Then, expectile regression minimizes the following squared pinball loss:

$$L_p^{als}(t) = \begin{cases} pt^2 & \text{if} \quad t \geq 0 \\ (1-p)t^2 & \text{if} \quad t < 0 \end{cases}$$

The expectile distance between two sets can be maximized by the following ALSSVM:

$$\begin{aligned} \min \ & \frac{1}{2}\|w\|^2 + \frac{C}{2}\sum_{i=1}^{m} L_p^{als}(\xi_i) \\ s.t. \quad & 1 - y_i(w.\phi(x_i) + b) = \xi_i, \quad i = 1, 2, \ldots, m \end{aligned} \tag{2.1}$$

where $\phi(x)$ is feature map. Then, Dual of (2.1) is written as

$$\begin{aligned} \min \ & \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \lambda_i\lambda_j y_i y_j K(x_i, x_j) + \frac{1}{2Cp}\sum_{i=1}^{m}\left(\lambda_i + \frac{1}{1-p}\beta_i\right)^2 \\ s.t. \quad & \sum_{i=1}^{m} \lambda_i y_i = 0, \\ & \lambda_i + \beta_i \geq 0, \ \ \beta_i \geq 0, \quad i = 1, 2, \ldots, m \end{aligned} \tag{2.2}$$

And the bias term b is computed according to

$$y_i\left(\sum_{j=1}^{m} y_j\lambda_j K(x_i, x_j) + b\right) = 1 - \frac{1}{p}\xi_i, \forall i : \alpha_i \geq 0$$

$$y_i\left(\sum_{j=1}^{m} y_j\lambda_j K(x_i, x_j) + b\right) = 1 + \frac{1}{1-p}\xi_i, \forall i : \beta_i \geq 0$$

3. Solving m-th order initial value problem by using ALSSVM model

A m-th order initial value problem the following are introduced as:

$$y^{(m)}(t) - \sum_{i=1}^{m} f_i(t) y^{(m-i)}(t) = r(t)$$
$$y(a) = p_1, \;\; y^{(i-1)}(a) = p_i, \;\; i = 1, 2, \ldots, m \;\; t \in [a, c] \tag{3.1}$$

Now, we uses the mentioned algorithm for solving m-th order linear ordinary differential equations and approximates them solution. Let us assume that a general approximate solution to (3.1) is of the form of $\bar{y}(t) = w.\phi(t)+b$, where w and b are unknowns; And we should specifies of them. Therefore, to obtain the optimal value of these parameters, we assume a discretization of the interval $[a, c]$ into a set of collocation points $\Delta = \{a = t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_N = c\}$.
In order to, by combining ALSSVM framework model cost function with constraints constructed by imposing the approximate solution for equation of (3.1) the following form constructed:

$$\begin{aligned}
&\min \;\; \frac{1}{2}\|w\|^2 + \frac{C}{2}\sum_{i=2}^{N}\xi_i^2 \\
&s.t. \quad w.\left(\phi^{(m)}(t_i) - \sum_{k=1}^{m} f_k(t_i)\phi^{(m-k)}(t_i)\right) - f_m(t_i)b - r(t_i) \leq \frac{\xi_i}{p}, \;\; i = 2, 3, \ldots, N \\
&\quad - w.\left(\phi^{m}(t_i) - \sum_{k=1}^{m} f_k(t_i)\phi^{(m-k)}(t_i)\right) + f_m(t_i)b + r(t_i) \leq \frac{\xi_i}{1-p}, \;\; i = 2, 3, \ldots, N \\
&\quad w.\phi(t_1) + b = p_1 \\
&\quad w.\phi^{(i-1)}(t_1) = p_i, \;\; i = 2, 3, \ldots, m
\end{aligned} \tag{3.2}$$

the dual of (3.2) is reformed in the following matrix form:

$$\begin{aligned}
&\min \;\; \frac{1}{2}X^T H X + FX \\
&s.t. \quad DX = 0, \\
&\quad\;\; AX \leq 0
\end{aligned}$$

The obtained approximation solution is as follows,

$$\hat{y}(t) = \sum_{i=2}^{N}\lambda_i\left([\nabla_m^0 K](t_i, t) - \sum_{k=1}^{m} f_k(t_i)[\nabla_{m-k}^0 K](t_i, t)\right) + \sum_{j=1}^{m}\gamma_j[\nabla_{j-1}^0 K](t_1, t) + b$$

4. Numerical results

Example 4.1. Consider the following second order IVP:

$$y''(t) + \frac{1}{5}ty'(t) + y(t) = \frac{-1}{5}\exp(-\frac{t}{5})\cos(t), \quad y(0) = 0, \quad y'(0) = 1, \ \ t \in [0, 4]$$

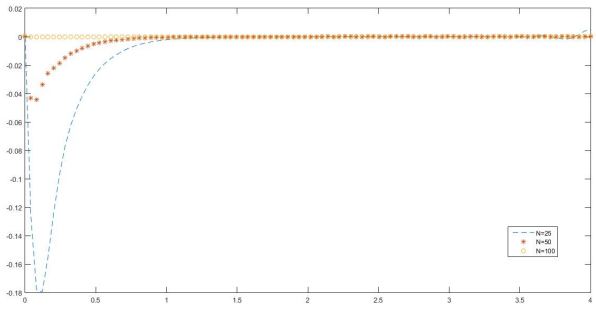

Figure 1. Numerical solutions with $N = 25, 50$ and 100

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A NEW OPTIMIZED TAYLOR ACCELERATED OVER-RELAXATION ITERATIVE METHOD FOR SOLVING LINEAR SYSTEMS

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Abstract. The Accelerated Over-Relaxation (AOR) iterative method is an iterative method for solving linear systems based on some parameters. Recently, some methods have been presented to optimize two of these parameters, to obtain better solution with lower time consumption. In this paper, by applying Taylor approximation, an exact optimization technique is proposed to find the optimal values of the parameters of the AOR iteration, and also to minimize the 2-norm of the residual vector. Meanwhile, numerical results show the efficiency of the presented optimal technique in contrast to the normal AOR iterative method.

1. Introduction

Lately, the iterative solvers of a large sparse linear system

$$Ax = b \tag{1.1}$$

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* Speaker.

has been regarded in many scientific computing and engineering problems, where the coefficient matrix $A \in R^{n\times n}$ is a nonsingular matrix, $b \in R^n$ is a given right-hand vector and $x \in R^n$ is an unknown vector. The accelerated overrelaxation (AOR) method, which has been verified to be a potent device for solving the linear system of equations (1.1), was offered firstly by Hadjidimos [1]. In particular, he displayed that when the two parameters are easily achieved the method converges faster than the other methods of the same type. Therefore, the gravity about the determination of the optimal acceleration and overrelaxation parameters has to be further perused. In addition, analytic formulas for optimal parameters were also presented by Hadjidimos [1, 3]. Thus, usages of the AOR method to widespread real problems are seriously restricted. The asymptotically optimal successive overrelaxation method of opting the optimal factor in a dynamic fashion according to known information at the current iterate step, was drafted by Bai and Chi [4]. Based on the mentioned facts, an optimization technique relating to choosing the optimal parameters is put forward. In [2], the optimal parameters ω and γ are computed by solving a lower-order nonlinear system that is determined by the residual vector and the coefficient matrix A. Furthermore, the optimal parameters are selected by the Newton iterative method instead of specific analytic formulas in [1, 3]. recently, Shi and Yu presented a new version of the AOR method [5]. Based on the AOR idea, in this study, we apply an exact optimization technique to present a new iterative method for solving (1.1); this method is more stable and effective than the method presented in [2]. In following, we briefly review the AOR method and its properties.

1.1. **AOR method and its attributes.** For solving (1.1), Hadjidimos [1] proposed the following splitting method with two parameters for the coefficient matrix A as $A = M_{\gamma,\omega} - N_{\gamma,\omega}$ where

$$M_{\gamma,\omega} = \frac{1}{\omega}(A_D - \gamma A_L),\ N_{\gamma,\omega} = \frac{1}{\omega}[(1-\omega)A_D + (\omega-\gamma)A_L + \omega A_u], \tag{1.2}$$

where $\gamma,\ \omega \neq 0$, A_D is the diagonal part of A, and $-A_L$ and $-A_u$ are strictly lower and strictly upper triangular parts of A, respectively. The iteration format of AOR method for solving the linear system (1.1) is

$$x^{p+1} = \ell_{\gamma,\omega}x^p + g_{\gamma,\omega}, \tag{1.3}$$

where

$$\ell_{\gamma,\omega} = (A_D - \gamma A_L)^{-1}[(1-\omega)A_D + (\omega-\gamma)A_L + \omega A_u];$$

$g_{\gamma,\omega} = \omega(A_D - \gamma A_L)^{-1}b.$

We apperceive the particular values of the parameters γ and ω in [1] when the AOR method can be simplified into
- the Jacobi method if $\omega = 0$, $\gamma = 1$;
- the Simultaneous Overrelaxation method if $\gamma = 0$;
- the Gauss-Seidel method if $\omega = 1$, $\gamma = 1$;
- the Successive Overrelaxation method if ω=γ.

More precisely, by scrutiny we have the following conclusion.

Theorem 1.1. *(Theorem 4.4 of [3]) Let A be a nonsingular matrix with nonzero diagonal entries. Then $\{x^p\}_{p=0}^{\infty}$ in (1.3) generated by the AOR method, converges to the unique solution x_* of the linear system (1.1) if A is a real symmetric positive definite matrix, $0 < \gamma < 2$ and $0 < \omega < \frac{2\gamma}{1+\rho(\ell_{\gamma,\gamma})}$ where ρ is the sprctral radius of the iteration matrix.*

2. Modified optimized Taylor AOR iterative method

Recently, AOR method has been modified by using an optimization technique. Since $(A_D - \gamma A_L)^{-1} = (I - \gamma L)^{-1}A_D^{-1}$ with $L = A_D^{-1}A_L$ is a strictly lower triangular matrix, $L^n = 0$ (the zero matrix) and $\rho(\gamma L) < 1$; also $(I - \gamma L)^{-1}$ can be written in the form of Taylor expansion. Then $(I - \gamma L)^{-1} = \sum_{k=0}^{n-1}(\gamma L)^k$ and thus $M_{\gamma,\omega}^{-1}$ can be expressed as

$$M_{\gamma,\omega}^{-1} = \omega(A_D - \gamma A_L)^{-1} = \omega(I - \gamma L)^{-1}A_D^{-1} = \omega\sum_{k=0}^{n-1}(\gamma L)^k A_D^{-1}. \quad (2.1)$$

Obviously, $M_{\gamma,\omega}^{-1}$ can be approximated by a lower-order truncation of the matrix series on the right-hand side of (2.1). Generally,

$$M_{\gamma,\omega}^{-1} = \omega(I + \alpha\gamma L + \beta^2\gamma^2 L^2)A_D^{-1}, \quad (2.2)$$

where α and β are two real parameters used in Taylor expansion of γL. Let ϵ^p denote the error vector of this modified AOR method at the pth iterate step, that is, $\epsilon^p = x^p - x_*$ and let $H_{\gamma,\omega} = AM_{\gamma,\omega}^{-1} - L$. So we have

$$H_{\gamma,\omega} = AM_{\gamma,\omega}^{-1} - I \simeq \omega A(I + \alpha\gamma L + \beta^2\gamma^2 L^2)A_D^{-1} - I. \quad (2.3)$$

Then we have the following theorem:

Theorem 2.1. *If A is a symmetric positive definite matrix, then*

$$\begin{aligned}\|\epsilon^{p+1}\|_A^2 &= (r^p)^T H_{\gamma,\omega}^T A^{-1} H_{\gamma,\omega} r^p \\ &\simeq (r^p)^T (A_D^{-1}(I + \alpha\gamma L + \beta^2\gamma^2 L^2)^T A\omega \\ &\quad - I)(\omega(I + \alpha\gamma L + \beta^2\gamma^2 L^2)A_D^{-1} - A^{-1})r^p.\end{aligned}$$

Proof.

$$\begin{aligned}\|\epsilon^{p+1}\|_A^2 &= (\epsilon^{p+1}, A\epsilon^{p+1}) = (\epsilon^p + M_{\gamma,\omega}^{-1} r^p, A\epsilon^p + AM_{\gamma,\omega}^{-1} r^p) \\ &= (A^{-1}(-r^p + AM_{\gamma,\omega}^{-1} r^p), -r^p + AM_{\gamma,\omega}^{-1} r^p) \\ &= (A^{-1} H_{\gamma,\omega} r^p, H_{\gamma,\omega} r^p) \\ &= (r^p)^T H_{\gamma,\omega}^T A^{-1} H_{\gamma,\omega} r^p.\end{aligned}$$

□

3. Main results

Using the results of Theorem 2.1 and relations (2.2) and (2.3), the following algorithm is proposed for solving iteratively (1.1). We remind that Theorem 1.1 guarantees the convergence of this algorithm for $2 > \omega \geq \gamma > 0$.

Proposed Algorithm

S0. Given an initial vector $x_0 \in R^n$, a precision ϵ_1, for $k = 0, 1, 2, ...$:

S1. Compute $r^k = b - Ax^k$.

S2. Solve $\min\limits_{2>\omega\geq\gamma>0} \|\epsilon^{p+1}\|_A$ by trust region method for obtaining $\gamma^k, \omega^k, \alpha^k, \beta^k$.

S3. Compute $x^{k+1} = x^k + \omega(I + \alpha\gamma L + \beta^2\gamma^2 L^2)A_D^{-1} r^k$.

Due to lack of space, numerical results will be expressed in the oral speech

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SOLUTION OF FRACTIONAL LINEAR OPTIMAL CONTROL SYSTEMS USING TRIANGULAR FUNCTIONS

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Abstract. In this paper, the operational matrix of triangular functions (TFs) for fractional order integration in the Caputo sense are applied to approximate the solutions of fractional optimal control of linear systems, which have a quadratic performance index. The necessary optimality conditions are stated in the form of fractional two-point boundary value problem, then this problem is converted to a coupled Volterra integral equations.

1. Introduction

Fractional optimal control problem (FOCP) is an optimal control problem in which the objective functional or the differential equations governing the dynamics of the system, contain at least one fractional derivative term. Agrawal proved a version of Euler-Lagrange equations for fractional problems of the calculus of variations in the sense of Caputo [1]. Also, after imposing the Pontryagin's minimum principle (PMP) to the considered FOCP, we obtain a fractional two-point boundary value problem [4]. In this paper, we consider the following fractional linear optimal control problem (FLOCP)

$$\begin{aligned} &\text{Min } J = \frac{1}{2}\int_0^1 (x^T P x + 2x^T Q u + u^T R u)dt \\ &s.t. \\ &\quad \begin{cases} {}_0D_t^\alpha x(t) = Ax(t) + Bu(t), \\ x(0) = x_0, \end{cases} \end{aligned} \tag{1.1}$$

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* Speaker.

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ and ${}_0D_t^\alpha$ stands for the α-th left Caputo fractional derivative with $0 < \alpha \leq 1$. When $\alpha = 1$, the above problem is converted to the standard linear optimal control problem. The control $u(t)$ is an admissible control if it is piecewise continuous in t for $t \in [0, 1]$. Its values belong to a given closed subset U of $\mathbb{R}^+$. The input control $u(t)$ is derived by minimizing the quadratic performance index J, where $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times m}$ are positive semi-definite matrices and $R \in \mathbb{R}^{m\times m}$ is positive definite matrix.

2. Optimality conditions

We consider Hamiltonian for system (1.1) as [3]

$$H(x, u, \lambda, t) = \frac{1}{2}(x^T Px + 2x^T Qu + u^T Ru) + \lambda^T(Ax(t) + Bu(t)), \quad (2.1)$$

where $\lambda \in \mathbb{R}^n$ is co-state vector. According to the Pontryagin's minimum principle, we have

$$\begin{cases} {}_0D_t^\alpha x(t) = \frac{\partial H}{\partial \lambda} = Ax(t) + Bu(t);\ x(0) = x_0, \\ {}_tD_1^\alpha \lambda(t) = \frac{\partial H}{\partial x} = Px + Qu + A^T\lambda;\ \lambda(1) = 0, \\ \frac{\partial H}{\partial u} = 0, \end{cases} \quad (2.2)$$

where the operator ${}_tD_1^\alpha$ indicates the right Caputo fractional derivative of order α. From the assumption $\frac{\partial H}{\partial u} = 0$, the exact optimal control is computed as:

$$u^* = -R^{-1}Q^T x - R^{-1}B^T\lambda. \quad (2.3)$$

Substituting (2.3) into (2.2) and taking the operators ${}_0I_t^\alpha$ and ${}_tI_1^\alpha$ from these equations, respectively, we obtain the following system

$$\begin{cases} x(t) = x_0 + {}_0I_t^\alpha\Big((A - BR^1Q^T)x(t) - (BR^{-1}B^T)\lambda(t)\Big), \\ \lambda(t) = \lambda_1 + {}_tI_1^\alpha\Big((P - QR^{-1}Q^T)x(t) + (-QR^{-1}B + A^T)\lambda(t)\Big). \end{cases} \quad (2.4)$$

3. Fractional operational matrices

Now we construct operational matrices of TFs for the right and left Riemann-Liouville fractional integrals of order α [2]. For the right fractional integration of the function $T1_i(t)$; $i = 0, 1, \cdots, m-1$, we have

$${}_tI_1^\alpha T1_i(t) = \begin{cases} -\frac{(ih-t)^\alpha((\alpha+i+1)h-t)-((i+1)h-t)^{\alpha+1}}{h\Gamma(\alpha+2)}, & 0 \leq t \leq ih, \\ \frac{((i+1)h-t)^{\alpha+1}}{h\Gamma(\alpha+2)}, & ih \leq t \leq (i+1)h, \\ 0, & t \geq (i+1)h. \end{cases} \quad (3.1)$$

Also, the right fractional integration of the function $T2_i(t)$ is

$$ {}_tI_1^{\alpha}T2_i(t) = \begin{cases} \frac{((i+1)h-t)^{\alpha}(\alpha-i+t)h+(ih-t)^{\alpha+1}}{h\Gamma(\alpha+2)}, & 0 \le t \le ih, \\ \frac{((\alpha-i)h+t)((i+1)h-t)^{\alpha}}{h\Gamma(\alpha+2)}, & ih \le t \le (i+1)h, \\ 0, & t \ge (i+1)h. \end{cases} \tag{3.2} $$

Expansion of ${}_tI_1^{\alpha}T1_i(t)$ with respect to TFs is:

$$ {}_tI_1^{\alpha}T1_i(t) \simeq [c_{i0}, \cdots, c_{im-1}]T1(t) + [d_{i0}, \cdots, d_{im-1}]T2(t), $$

where $c_{ij} = {}_tI_1^{\alpha}T1_i(jh)$ and $d_{ij} = {}_tI_1^{\alpha}T1_i((j+1)h)$, $i, j = 0, 1, \cdots, m-1$. From Eq. (2.4), we get

$$ \begin{cases} c_{ij} = 0; \ j \ge i+1 \\ c_{ij} = \frac{h^{\alpha}}{\Gamma(\alpha+2)}\Big((i-j+1)^{\alpha+1} - (i-j)^{\alpha}(\alpha+1) - (i-j)^{\alpha+1}\Big); \ j \le i \\ d_{ij} = c_{ij+1}. \end{cases} $$

Finally, for $i, j = 0, 1, \cdots, m-1$, we can write:

$$ {}_tI_1^{\alpha}T1(t) \simeq R1_{\alpha}T1(t) + R2_{\alpha}T2(t), \tag{3.3} $$

where $R1_{\alpha}$ and $R2_{\alpha}$ can be computed as

$$ R1_{\alpha} = \begin{pmatrix} \zeta_0 & 0 & 0 & \cdots & 0 \\ \zeta_1 & \zeta_0 & 0 & \cdots & 0 \\ \zeta_2 & \zeta_1 & \zeta_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \zeta_{m-1} & \zeta_{m-2} & \zeta_{m-3} & \cdots & \zeta_0 \end{pmatrix}, R2_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \zeta_0 & 0 & 0 & \cdots & 0 \\ \zeta_1 & \zeta_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \zeta_{m-2} & \zeta_{m-3} & \zeta_{m-4} & \cdots & 0 \end{pmatrix}, $$

where $\zeta_r = \frac{h^{\alpha}}{\Gamma(\alpha+2)}\Big((r+1)^{\alpha+1} - r^{\alpha}(\alpha+1) - (r)^{\alpha+1}\Big)$, $r = 0, 1, \cdots, m-1$.

In a similar manner, the following approximation can be achieved for $T2(t)$,

$$ {}_tI_1^{\alpha}T2_i(t) \simeq R3_{\alpha}T1(t) + R4_{\alpha}T2(t), \tag{3.4} $$

where $R3_{\alpha}$ and $R4_{\alpha}$ are as

$$ R3_{\alpha} = \begin{pmatrix} \xi_0 & 0 & 0 & \cdots & 0 \\ \xi_1 & \xi_0 & 0 & \cdots & 0 \\ \xi_2 & \xi_1 & \xi_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_{m-1} & \xi_{m-2} & \xi_{m-3} & \cdots & \xi_0 \end{pmatrix}, R4_{\alpha} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \xi_0 & 0 & 0 & \cdots & 0 \\ \xi_1 & \xi_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_{m-2} & \xi_{m-3} & \xi_{m-4} & \cdots & 0 \end{pmatrix}, $$

in which $\xi_r = \frac{h^{\alpha}}{\Gamma(\alpha+2)}\Big(r^{\alpha+1} + \alpha(r-1)^{\alpha} - r(r-1)^{\alpha}\Big)$, $r = 0, 1, \cdots, m-1$.

By using Eqs. (3.3) and (3.4), right fractional integration of $T(t)$ can be computed as:

$$\begin{aligned} {}_tI_1^{\alpha}T(t) &= \begin{pmatrix} {}_tI_1^{\alpha}T1(t) \\ {}_tI_1^{\alpha}T2(t) \end{pmatrix} \simeq \begin{pmatrix} R1_{\alpha}T1(t) + R2_{\alpha}T2(t) \\ R3_{\alpha}T1(t) + R4_{\alpha}T2(t) \end{pmatrix} \\ &= \begin{pmatrix} R1_{\alpha} & R2_{\alpha} \\ R3_{\alpha} & R4_{\alpha} \end{pmatrix} \begin{pmatrix} T1(t) \\ T2(t) \end{pmatrix} = R_{\alpha}T(t). \end{aligned} \tag{3.5}$$

Therefore, the right fractional integration of the arbitrary function $f(t)$ can be approximated as:

$$ {}_tI_1^{\alpha}f(t) \simeq F^T R_{\alpha}T(t). $$

In a similar fashion, we can compute the left fractional integration operational matrix, denoted L_{α}, of the function $T(t)$. Therefore, for any arbitrary function $f(t)$, the left fractional integration can be approximated as:

$$ {}_0I_t^{\alpha}f(t) \simeq F^T L_{\alpha}T(t). $$

In reference [2], the left fractional matrix L_{α} is given.

4. Approximation method

Consider the weakly singular system (2.4). Expanding the functions $x(t)$ and $\lambda(t)$ with respect to TFs, and applying the left and right operational matrices, we get the following system:

$$\begin{cases} X_0 + (A - BR^{-1}Q^T)XL_{\alpha} - X - (BR^{-1}B^T)\Lambda L_{\alpha} = 0 \\ \Lambda_1 + (P - QR^{-1}Q^T)XR_{\alpha} + (-QR^{-1}B + A^T)\Lambda R_{\alpha} - \Lambda = 0. \end{cases}$$

After solving the above linear matrix system in terms of the unknown coefficients of the vectors X and Λ, the state function $x_m(t)$ and the optimal control $u_m(t)$ will be obtained.

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CONVERGENCE ANALYSIS OF DISCRETE-TIME FRACTIONAL ORDER SYSTEMS USING CONTRACTION METHOD

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Abstract. This article describes contraction theory for the case of discrete-time fractional order systems. The proposed approach is useful for analyzing the stability of nonlinear and discrete-time fractional order systems. Furthermore, it leads to a significant conceptual simplification.

1. Introduction

Stability analysis has been applied to particular classes of fractional order systems [1], but it still lacks generality. In this paper, a new result is derived using elementary tools from continuum mechanics and differential geometry, leading to a theory which is called contraction analysis. Intuitively, contraction analysis is based on a slightly different view of what stability is, and it is motivated by the elementary

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* Speaker.

remark that talking about stability does not require to know what the nominal motion is. Intuitively, a system is stable in some region if initial conditions or temporary disturbances are somehow "forgotten", i.e., if the final behavior of the system is independent of the initial conditions [2, 4, 5]. All trajectories then converge to the nominal motion.

In this article we describe Contraction Theory for the case of discrete-time fractional order systems (DFOS).

Consider the following integer order discrete-time system:

$$\mathrm{x}\left(k+1\right)=\mathrm{f}(\mathrm{x}(k)) \tag{1.1}$$

where $\mathrm{f}(\cdot)$ is a smooth nonlinear vector function. A general DFOS can be represented as follows, which for more details the reader is referred to [3]:

$$\mathrm{x}\left(k+1\right)=\mathrm{f}\left(\mathrm{x}\left(k\right)\right)+\left(\alpha-1\right)\mathrm{x}\left(k\right)+\sum_{p=1}^{L}C_{p}\mathrm{x}(k-p) \tag{1.2}$$

where $C_{p}=(-1)^{p}\binom{\alpha}{p+1}$.

2. Main results

Theorem 2.1. *Exponential convergence of system (1.2) is guaranteed if*

$$\frac{\partial \mathrm{g}_{k}}{\partial \mathrm{x}\left(k\right)}^{T}\frac{\partial \mathrm{g}_{k}}{\partial \mathrm{x}\left(k\right)}-\mathrm{I} \tag{2.1}$$

be uniformly negative definite, where

$$\mathrm{g}_{k}=\mathrm{f}\left(\mathrm{x}\left(k\right)\right)+\left(\alpha-1\right)\mathrm{x}\left(k\right)+\sum_{p=1}^{L}C_{p}\mathrm{x}(k-p)\ . \tag{2.2}$$

Proof. The associated virtual dynamics of (1.2) is

$$\delta\mathrm{x}\left(k+1\right)=\frac{\partial \mathrm{g}_{k}}{\partial \mathrm{x}\left(k\right)}\delta\mathrm{x}\left(k\right) \tag{2.3}$$

so that the virtual length dynamics is

$$\delta\mathrm{x}^{T}\left(k+1\right)\delta\mathrm{x}\left(k+1\right)=\delta\mathrm{x}^{T}\left(k\right)\frac{\partial \mathrm{g}_{k}}{\partial \mathrm{x}\left(k\right)}^{T}\frac{\partial \mathrm{g}_{k}}{\partial \mathrm{x}\left(k\right)}\delta\mathrm{x}\left(k\right)\ , \tag{2.4}$$

therefore, the rate of change of the left hand side is

$$\delta\mathrm{x}^{T}\left(k+1\right)\delta\mathrm{x}\left(k+1\right)-\delta\mathrm{x}^{T}\left(k\right)\delta\mathrm{x}\left(k\right)$$

$$=\delta\mathrm{x}^{T}\left(k\right)\frac{\partial \mathrm{g}_{k}}{\partial \mathrm{x}\left(k\right)}^{T}\frac{\partial \mathrm{g}_{k}}{\partial \mathrm{x}\left(k\right)}\delta\mathrm{x}\left(k\right)-\delta\mathrm{x}^{T}\left(k\right)\delta\mathrm{x}\left(k\right)$$

$$= \delta \mathrm{x}^T(k) \left[\frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)}^T \frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)} - \mathrm{I} \right] \delta \mathrm{x}(k)$$

thus, exponential convergence to a single trajectory is guaranteed if

$$\frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)}^T \frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)} - \mathrm{I} \tag{2.5}$$

be uniformly negative definite. □

Corollary 2.2. *For the linear DFOS*

$$\mathrm{x}(k+1) = \mathrm{A}\mathrm{x}(k) + (\alpha - 1)\mathrm{x}(k) + \sum_{p=1}^{L} C_p \mathrm{x}(k-p) \tag{2.6}$$

exponential convergence to a single trajectory is guaranteed if $\mathrm{B}^T\mathrm{B} - \mathrm{I}$ *be uniformly negative definite, where* $\mathrm{B} = \mathrm{A} + (\alpha - 1)\mathrm{I}$.

Proof. We have $\frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)} = A + (\alpha - 1)\mathrm{I}$, which concludes the proof. □

Using the generalized virtual displacement

$$\delta \mathrm{z}(k) = \Theta_k(\mathrm{x}(k), k)\, \delta \mathrm{x}(k) \tag{2.7}$$

obtained from an invertible coordinate tranformation Θ and by relation (2.3) we have:

$$\begin{aligned} \delta \mathrm{z}^T(k+1)\, \delta \mathrm{z}(k+1) &= \delta \mathrm{x}^T(k) \frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)}^T \Theta_{k+1}^T \Theta_{k+1} \frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)} \delta \mathrm{x}(k) \\ &= \delta \mathrm{z}^T(k) F_k^T F_k \delta \mathrm{z}(k) \end{aligned} \tag{2.8}$$

where

$$F_k = \Theta_{k+1} \frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)} \Theta_k^{-1} \tag{2.9}$$

is the *discrete-time fractional order generalized Jacobian*. Now, we can give the following generalized definition of contraction region for DFOS.

Definition 2.3. Given the DFOS $\mathrm{x}(k+1) = \mathrm{g}_k(\mathrm{x}(k))$, with g_k given in (2.2), a region of the state space is called a contraction region with respect to a uniformly positive definite metric $\mathrm{M}_k(x(k), k) = \Theta_k^T \Theta_k$ if in that region

$$\exists \beta > 0 \quad , \quad F_k^T F_k - I \leq -\beta \mathrm{I} < 0$$

where $F_k = \Theta_{k+1} \frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)} \Theta_k^{-1}$.

Example 2.4. Consider the Logistic DFOS [3]:

$$\mathrm{x}(k+1) = \mu \mathrm{x}(k)(1 - \mathrm{x}(k)) + (\alpha - 1)\mathrm{x}(k) + \sum_{p=1}^{L} C_p \mathrm{x}(k-p).$$

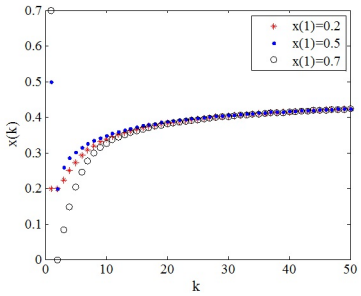

FIGURE 1. Convergence of Logistic DFOS for three arbitrary initial points 0.2, 0.5 and 0.7.

The fixed (equilibrium) point will be $\mathrm{x}^*=\frac{2-\mu-\alpha\pm\sqrt{(\mu+\alpha-2)^2+4\mu c}}{-2\mu}$ where $c=\sum_{p=1}^{L} C_p \mathrm{x}(k-p)$. Let

$$\mathrm{g}_k=\mu \mathrm{x}(k)(1-\mathrm{x}(k))+(\alpha-1)\,\mathrm{x}\,(k)+\sum_{p=1}^{L} C_p \mathrm{x}(k-p)$$

therefore, $\frac{\partial \mathrm{g_k}}{\partial \mathrm{x(k)}}=\mu+\alpha-1-2\mu \mathrm{x}\,(\mathrm{k})$ and the fractional order discrete-time Logistic system will be convergent if $\frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)}^T \frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)} - \mathrm{I} < 0$. Numerically, choosing $\alpha = 0.4$, $\mu = 2$ and $L = 50$, the fixed point will be $x^* = 0.4228$ and $(\frac{\partial \mathrm{g}_k}{\partial \mathrm{x}(k)})^2 - 1 = -0.9151 < 0$, therefore, the system will be converge to $x = 0.4228$ for each arbitrary initial point (see Fig.1).

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A GRADIENT-BASED NEURAL NETWORK METHOD FOR SOLVING SEMIDEFINITE PROGRAMMING PROBLEMS

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Abstract. Linear semidefinite programming (SDP) problems have received a lot of attentions because of large variety of applications. Motivated by some results for SDP complementarity problems, this paper gives a gradient-based neural network method for solving SDP. It is shown that the proposed neural network is asymptotically stable and converges to an exact optimal solution of the SDP problem.

1. Introduction

The SDP problem is given as below:

$$\text{minimize } f(x) = c^T x \tag{1.1}$$

$$\text{subject to } \quad A(x) = A_0 + \sum_{i=1}^{m} x_i A_i \preceq 0, \tag{1.2}$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{m\times m}, i = 1, 2, ..., n$ and $\preceq$ denotes the negative semidefinite order. The dual of SDP problem, DSDP, can be defined [2], as:

$$\begin{aligned} &\text{maximize } \langle A_0, Z\rangle \\ &\text{subject to } \langle -A_i, Z\rangle = c_i, \quad i = 1, 2, ...n, \\ &\qquad\qquad Z \succeq 0, \end{aligned}$$

Key words and phrases. Gradient-based neural network, Semidefinite programming, Convergent, Stability.

* Speaker.

The SDP problem is a convex optimization problem since its objective function and constraint are convex.

Theorem 1.1. *(Weak Duality Theorem) For every x feasible in SDP and every Z feasible in DSDP, we have*

(a) $c^T x - \langle A_0, Z\rangle \geq 0$
(b) *If $c^T x = \langle A_0, Z\rangle$, then x is optimal in SDP and Z is optimal in DSDP.*

Unlike linear programming, SDP having an optimal solution does not guarantee the same for its dual DSDP.

Theorem 1.2. *(Strong Duality Theorem) Suppose SDP and DSDP both are strictly feasible. Then SDP and DSDP both have optimal solutions and their optimum objective values coincide.*

Let $\mathcal{V} = \mathbb{R}^n \times S^m$. For a given $v \in \mathcal{V}$, we use the following notations for simplicity $v = (x, Z)$, where $x \in \mathbb{R}^n$ and $Z \in S^m$, respectively. For a given matrix $A \in S^m_+$, $A^{\frac{1}{2}} \in S^m_+$, denotes the matrix such that $A = A^{\frac{1}{2}}A^{\frac{1}{2}}$. Note that $A^{\frac{1}{2}} = P^T diag[\lambda_1^{\frac{1}{2}}, \ldots, \lambda_m^{\frac{1}{2}}]P$, where λ_j denotes jth eigenvalue of A in nonincreasing order and P is the matrix in the spectral decomposition $A = P^T diag[\lambda_1, \ldots, \lambda_m]P$. The operator $vec(.) : M_{m,n} \to \mathbb{R}^{mn}$ will be used to explicitly transform matrices into vectors by stacking the columns on top of each other. The notation $DA(x)$ represents the gradient of the mapping $A(.)$ at x. Thus, $DA(x)$ is a linear operator from $\mathbb{R}^n$ into S^m defined by $DA(x)y = \sum_{i=1}^n y_i \frac{\partial A(x)}{\partial x_i}$ and $\frac{\partial A(x)}{\partial x_i} = A_i$. Finally, we define the adjoint operator V^* the formula

$$V^* Z = (\langle V_1, Z\rangle, ..., \langle V_n, Z\rangle)^T, \quad \forall V \in S^m. \tag{1.3}$$

Note also that a gradient of $\langle A(x), Z\rangle$ with respect to x is given by

$$\nabla_x \langle A(x), Z\rangle = DA(x)^* Z. \tag{1.4}$$

1.1. **A Gradient neural network model.** Let us define the Lagrangian function of SDP by

$$L(y) = f(x) + \langle A(x), Z\rangle,$$

where $y = (x, Z)$, and $Z \in S^m$ is the Lagrange multiplier matrix for the positive semidefiniteness constraint. Then the Karush-Kuhn-Tucker (KKT) conditions for optimality of SDP are given by the following:

$$\nabla L_x(y) = 0, \tag{1.5}$$

$$\langle Z, A(x)\rangle = 0, \tag{1.6}$$

$$A(x) \in S^m_+, \qquad Z \in S^m_+. \tag{1.7}$$

where$\nabla L_x(y) = \nabla f(x) + DA(x)^* Z$.

Assumption 1: The matrices $A_i, i = 1, ..., n$ are linearly independent.

Assumption 2 :(Slater condition) Both the primal and the dual problems are strictly feasible.

Assumption 2 implies that the duality gap $\langle A(x), Z\rangle = 0$ for optimal solutions (x, Z).By this assumption, it is well known that x_* is a solution of SDP if and only if there exists a Z_* such that (x_*, Z_*) satisfying the KKT conditions (1.5)-(1.7). Now let us define a mapping $\Phi : S^m \times S^m \to S^m$

$$\Phi(X, Y) = (X^2 + Y^2)^{\frac{1}{2}} - (X + Y),$$

which is obvious an extension of the definition of Fischer-Burmeister function, with the arguments being symmetric matrices rather than two real numbers.

Lemma 1.3. *Let $X, Y \in S^m_+$. Then $XY = 0$ if and only if $\langle X, Y\rangle = 0$.*

Lemma 1.4. *Tseng [30, Lemma 6.1]*

$$\Phi(X, Y) = 0 \iff X \in S^m_+,\ Y \in S^m_+,\ XY = 0.$$

By Lemma 1.3 and lemma 1.4, (1.6) and (1.7) can be rewrite as $\Phi(A(x), Z) = 0$. We now generalize the smoothed Fischer-Burmeister function Φ_μ in an obvious way: Define $\Phi_\mu : S^m \times S^m \to S^m$ by

$$\Phi_\mu(X, Y) = (X^2 + Y^2 + 2\mu^2 I)^{\frac{1}{2}} - (X + Y).$$

Then we can state the following result.

Lemma 1.5. *Let $\mu > 0$ be any positive number, and let Φ be defined by (2.7). Then*

$$\Phi_\mu(X, Y) = 0 \iff X \in S^m_+,\ Y \in S^m_+,\ XY = \mu^2 I.$$

Definition 1.6. For any $C \in S^m_+$, define the linear mapping $\mathcal{L}_C : S^m \to S^m$ by

$$\mathcal{L}_C[X] := CX + XC.$$

Lemma 1.7. *Fix any $\mu > 0$ and any $X, Y, U, V \in S^m$. For Φ_μ given by (7), we have that Φ_μ is Frechet-differentiable and*

$$\nabla\Phi_\mu(X, Y)(U, V) = U + V - \mathcal{L}_C^{-1}[XU + UX + YV + VY],$$

where $C = (X^2 + Y^2 + 2\mu I)^{\frac{1}{2}}$.

Let $z = vec(Z)$, $a^i = vec(A_i)$ $(i = 1, ...n)$, $a(x) = vec(A(x))$, $\varphi_\mu(a(x), z)) = vec(\Phi_\mu(A(x), Z)$ and $\mathcal{A} = \begin{bmatrix} a^1 & a^2 & \dots & a^n \end{bmatrix}^T$.

Lemma 1.8. *If $y_* = (x_*^T, z_*^T)^T$ satisfies the following equation*

$$\eta(y) = \begin{bmatrix} -\big(\nabla f(x) + \mathcal{A}\big[\varphi_\mu(a(x), z) + z\big]\big) \\ \varphi_\mu(a(x), z) \end{bmatrix} = 0, \tag{1.8}$$

then x_* is a KKT point of SDP for every $\mu \to 0_+$.

Now we can easily verify that for find the KKT system (1.8) we can solve the following unconstrained smooth minimization problem:

$$minimize\ E(y) = \frac{1}{2}\|\eta(y)\|^2$$

let $x(.)$ and $Z(.)$, be some time dependent variables. We can use the steepest descent method to construct the following neural network model for solving SDP problem (1)-(2) as:

$$\frac{dy(t)}{dt} = -\rho\nabla E\big(y(t)\big),\ \rho > 0, \tag{1.9}$$

$$y(0) = y_0, \tag{1.10}$$

where ρ is a scale parameter.

1.2. **Stability and Convergence Properties.**

Theorem 1.9. (a) *If x_* is a solution to the SDP, then $(x_*^T, z_*^T)^T$ is an equilibrium point of (1.9) and (1.10), where Z_* is the Lagrange multiplier associated with x_*.*

(b) *If the Jacobian matrix $\nabla\eta(y)$ of the mapping η defined in (1.8) be nonsingular and $y_* = (x_*^T, z_*^T)^T$ is an equilibrium point of (1.9) and (1.10), Then x_* is an solution to the SDP.*

Theorem 1.10. *The equilibrium point of the proposed neural network model (1.9) and (1.10) is unique.*

Theorem 1.11. *Let y_* be an isolated equilibrium point of (1.9) and (1.10). Then y_* is asymptotically stable for (1.9) and (1.10)*

Theorem 1.12. *Suppose that $y = y(t, y_0)$ is a trajectory of (1.9) and (1.10) in which the initial point is $y_0 = y(0, y_0)$ and the level set $L(y_0) = \{y \in \mathbb{R}^{n+m^2} : E(y) \le E(y_0)\}$ is bounded, then a) $\gamma^+ = \{y(t, y_0) \mid t \ge 0\}$ is bounded; and b) There exists $\bar{y}$ such that $\lim_{t\to\infty} y(t, y_0) = \bar{y}$.*

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AN APPROXIMATE SOLUTION OF HIV INFECTION MODEL

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Abstract. In this paper, by means of the optimal control and power series, the optimal control power series technique is presented. By this method numerical solutions of the HIV infection model of CD4^+T cells are obtained.

1. Introduction

In this paper, the optimal control power series technique for finding the numerical solution of an epidemic model is presented in four steps. To explain these fundamental steps in optimal control power series technique, consider a nonlinear system as

$$\dot{x} = F(x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $x \in \mathbf{R}^n$.
Step 1 We can formulate this system as an optimal control problem for optimization. The general form is

$$\dot{x} = F(x(t), u(t)), \quad x(t_0) = x_0, \tag{1.2}$$

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* Speaker.

where $u(t) \in \mathbf{R}^m$ is control vector.
Step 2 The aim of the State-Dependent Riccati Equation (SDRE) control is to determine the sub-optimal controller for the system (1.2) such that the following coast functional is minimized:

$$\begin{aligned} J &= \int_{t_0}^{\infty} (x(t)^T\, Q\, x(t) + u(t)^T\, R\, u(t))dt, \\ s.t. \quad \dot{x} &= f(t, x(t)) + g(t, x(t))\, u(t), \end{aligned} \tag{1.3}$$

where $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{m\times m}$ are state dependent weighting matrices which satisfying $Q \geq 0$ and $R > 0$ for all x.
Step 3 According to the Pontryagin's maximum principle, the optimality conditions for (1.3) are determined by the following nonlinear two-point boundary value problem (TPBVP):

$$\begin{aligned} \dot{x} &= f(t, x(t)) + g(t, x(t)) \left[-R^{-1}\, g^T(t, x(t))\, \lambda(t)\right], \\ \dot{\lambda} &= -\left(Q\, x(t) + (\tfrac{\partial f(t,x(t))}{\partial x})^T\, \lambda(t) + \sum_{i=1}^{n} \lambda_i \left[-R^{-1}\, g^T(t, x(t))\, \lambda(t)\right]^T\, \tfrac{\partial g_i(t,x(t))}{\partial x} \right) \end{aligned} \tag{1.4}$$

where $\lambda(t) \in \mathbf{R}^n$. On the other hand, the optimal control law is illustrated by $u(t)^* = -R^{-1}\, g^T(t, x(t))\, \lambda(t)$.
Step 4 The system (1.4) contains a nonlinear TPBVP that cannot be solved analytically. But a solution can be expressed in terms of a power series which takes the form

$$x(t) = \sum_{n=0}^{\infty} c_n\, (t - t_0)^n, \qquad \lambda(t) = \sum_{n=0}^{\infty} d_n\, (t - t_0)^n, \tag{1.5}$$

for some fixed t_0. Substituting the power series into the system (1.4) gives some relationships among the coefficients $\{c_n\}$, which gives a power series solution.

2. Numerical application

In this section, we will apply the optimal control power series technique to a dynamical model of HIV CD4$^+$T cells [1]

$$\begin{aligned} \frac{dT}{dt} &= q - \alpha\, T + r\, T \left(1 - \frac{T + I}{T_{\max}}\right) - k\, V\, T, \\ \frac{dI}{dt} &= k\, V\, T - \beta\, I, \quad , \quad \frac{dV}{dt} = \mu\, \beta\, I - \gamma\, V, \end{aligned} \tag{2.1}$$

with the initial conditions: $T(0) = T_0, I(0) = I_0$ and $V(0) = V_0$. Here $T(t)$ represents the concentration of healthy CD4$^+$T cells at time t, $I(t)$ represents the concentration of infected CD4$^+$T cells at time t, and $V(t)$

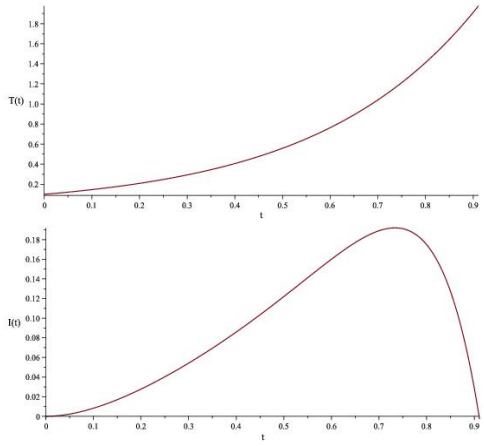

FIGURE 1. Graphic of the approximate solutions of T(t) and I(t).

represents the concentration of free HIV at time t. In recent years, several effective methods, including the Laplace Adomian decomposition method, the Homotopy perturbation method, the Bessel collocation method, the variational iteration method have been recommended to find the approximated solutions of HIV CD4$^+$T cells. Throughout this section, $q = 0.1, \alpha = 0.02, \beta = 0.3, r = 3, \gamma = 2.4, k = 0.0027, N = 10$ and $T_{\max} = 1500$. By optimal control power series technique, the following approximated solutions are obtained:

$$\begin{aligned} T &= 0.1 + 0.3977800000\,t + 0.5922148450\,t^2 + 0.5875974940\,t^3 \\ &\quad + 0.4370467812\,t^4 + 0.2598795654\,t^5 + 0.1284691841\,t^6 + \cdots \\ I &= 0.1225420020000\,t + 0.9993696680\,t^2 - 2.120992477\,t^3 \\ &\quad + 3.799090900\,t^4 - 4.597792247\,t^5 + 4.911909967\,t^6 + \cdots \\ V &= 0.1 - 0.2400000000\,t + 0.1880213003\,t^2 - 0.05048007343\,t^3 \\ &\quad - 0.1601232001\,t^4 + 0.2882108934\,t^5 - 0.3384965683\,t^6 + \cdots \end{aligned} \tag{2.2}$$

3. Conclusions and Discussions

In this paper, numerical solutions of a model for HIV infection of CD4$^+$T cells are obtained. The solutions of $T(t)$ and $I(t)$ for $0 \leq t \leq 0.9$, as shown in Fig 1 are plotted. Fig. 1 shows that, by applying optimal control the number of $T(t)$ increases gradually. In Fig 2, after introducing control variable $u_I(t)$, the density of the concentration of $I(t)$ declines towards zero. In Tables 1-2, the obtained values of the

approximate solutions of a model for HIV infection of CD4$^+$T cells at several values of t are compared with those of Laplace Adomian decomposition method with Pade approximation [2] and Bessel collocation method [3]. In Table 1, it is found that the solutions which are obtained by applying our present method are in a good agreement with the approximated solutions in [2] and [3] at $0 \leq t \leq 0.6$. Also, it can be concluded that our present results at $0.6 \leq t \leq 0.9$, even better than the results obtained by the Laplace Adomian decomposition method with Pade approximation [2] and Bessel collocation method [3]. The results in Table 2 show that, in [2] and [3] the density of infected cells $I(t)$ increase at $0 \leq t \leq 0.9$. But in our present method, the intensity of infected cells $I(t)$ decrease with the passage of time after applying control variable $u_I(t)$.

TABLE 1. Numerical comparison for $T(t)$.

t	$LADM - Pade$	*Bessel coll.*	*Present method*
0	0.1	0.1	0.1
0.2	0.2088072731	0.2038616561	0.2087367817
0.4	0.4061052625	0.3803309335	0.4059521352
0.6	0.7611467713	0.6954623767	0.7635498323
0.9	1.5245154522	1.4521254658	1.977214845

TABLE 2. Numerical comparison for $I(t)$.

t	$LADM - Pade$	*Bessel coll.*	*Present method*
0	0	0	0
0.2	0.0000060327	0.0000062478	0.02772720781
0.4	0.0000131591	0.0000129355	0.08586776364
0.6	0.0000212683	0.0000203526	0.16088182680
0.9	0.0000385462	0.0000325452	0.00000600060

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A STABILIZER CONTROL FOR NONLINEAR SYSTEMS

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Abstract. In this paper, an approach is presented to design a stabilizer control for nonlinear systems. The hyperbolic systems are first introduced and then a stabilizer control is designed. This stabilizer control can be utilized for any nonlinear control system. By applying this control for an inverted pendulum system, the efficiency of approach is illustrated.

1. Introduction

Many applied systems in the world are nonlinear and designing control to these systems is usually a full challenging work. In recent years, an approach based on the fuzzy rule-base systems has been presented which is called hyperbolic Modeling (see [1, 2, 3] for details). In this work, we show that we can achieve to the hyperbolic model corresponding to any nonlinear control system without utilizing the fuzzy concepts. Also, we show that this hyperbolic model can be easily obtained by solving an optimization problem. Here, we propose a stabilizer control to the hyperbolic system and apply it for the main nonlinear

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Key words and phrases. Hyperbolic model, Control systems, Stability.

* Speaker.

system. By applying the approach for an inverted pendulum system, we show the efficiency of the approach.

2. Hyperbolic model

Consider the following nonlinear control system

$$\begin{cases} \dot{x} = A\tanh(kx) + Bu, \\ x(0) = \alpha, \end{cases} \tag{2.1}$$

where $x = (x_1, \dots, x_n)$ and $u = (u_1, \dots, u_p)$ are the state and control variables, respectively, $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times p}$ are constant matrices, $k = (k_1, \dots, k_n)^T$ and $\alpha = (\alpha_1, \dots, \alpha_n)^T$ are constant vectors, and $\tanh(kx) = \big(\tanh(k_1x_1), \dots, \tanh(k_nx_n)\big)^T$. Control system (2.1) is called a hyperbolic model.

2.1. Design a stabilizer control.

A control to stabilize the nonlinear control system (2.1) can be given as follows

$$u = H\tanh(kx) \tag{2.2}$$

where $H = (h_{ij})_{n\times n}$ is a constant matrix. By replacing control (2.2) in system (2.1), we get the following system

$$\begin{cases} \dot{x} = (A + BH)\tanh(kx), \\ x(0) = \alpha. \end{cases} \tag{2.3}$$

Theorem 2.1. *(See [1, 2]) Consider system (2.3). If there exists a diagonal matrix $p > 0$ such that*

$$P(A + BH) + (A + BH)^T P + I = 0, \tag{2.4}$$

then system (2.3) is globally asymptotically stable $\big($i.e. for any $\alpha \in \mathbb{R}^n$, $\lim_{t\to\infty} x(t) = 0\big)$

3. Constructing the hyperbolic from of nonlinear systems

Consider the following nonlinear control system

$$\begin{cases} \dot{x} = f(x, u), \\ x(0) = \alpha, \end{cases} \tag{3.1}$$

where $f(\cdot, \cdot)$ is a continuous function. We suggest the following optimization problem to achieve the hyperbolic form of the system (3.1):

$$\underset{A,B,k}{\text{Minimize}} \quad J(A, B, k) = \int_{N_\delta(0)} \int_{N_\delta(0)} \big(A\tanh(kx) + Bu - f(x, u)\big)^2 dxdu \tag{3.2}$$

where $N_\delta(0)$ is a neighborhood of origin (i.e. the equilibrium point of system (3.2)) and $\delta > 0$ is a small number. The optimization problem (3.2), can be easily solved by approximate discretization methods.

4. Numerical simulation

Consider the following inverted pendulum system (see Figure 1(a))

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = F(x_1, x_2) + G(x_1, x_2)u, \\ x_1(0) = \alpha_1, \quad x_2(0) = \alpha_2, \end{cases} \tag{4.1}$$

where

$$F(x_1, x_2) = \frac{g(m_c + m) - mlx_2^2 \cos x_1 \sin x_1}{l(m_c + m)(\frac{4}{3} - \frac{m \cos^2 x_1}{m_c + m})},$$
$$G(x_1, x_2) = \frac{\cos x_1}{l(m_c + m)(\frac{4}{3} - \frac{m \cos^2 x_1}{m_c + m})}.$$

Here, g is the acceleration of gravity, m_c is the mass of cart, m is the mass of the pole, $2l$ is the pole's length and u is the applied force (or control). We assume that $m = 0.1$, $m_c = 1$ and $l = 0.5$. We first use the following coordinate transformation and then apply the presented approach:

$$y_1 = x_1, \quad y_2 = x_1 + x_2.$$

By these, we convert the system (4.1) into the following system

$$\begin{cases} \dot{y_1} = y_2 - y_1, \\ \dot{y_2} = F(y_1, y_2 - y_1) + G(y_1, y_2 - y_1)u, \\ y_1(0) = \alpha_1, \quad y_2(0) = \alpha_1 + \alpha_2. \end{cases}$$

Now, we solve the corresponding optimization problem (3.2) for $\delta = 0.1$. Here, we apply the trapezoidal formula, for $N = 20$ points, to discretize the integral. By solving the obtained nonlinear programming problem, we get

$$A = \begin{bmatrix} -0.9822 & 0.9089 \\ 14.48 & 1.1765 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1.4603 \end{bmatrix}, \quad k = \begin{bmatrix} 1.0204 \\ 0.8513 \end{bmatrix}.$$

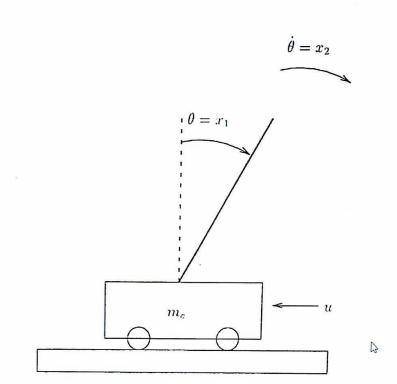
We assume that $P = diag(P_1^2, P_2^2)$ and $H = [h_1, h_2]^T$, and solve the corresponding system (2.4). We get

$$P = \begin{bmatrix} P_1^2 & 0 \\ 0 & P_2^2 \end{bmatrix} = \begin{bmatrix} 0.7135 & 0 \\ 0 & 0.1385 \end{bmatrix}, \quad H = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -31.2898 \\ -18.6460 \end{bmatrix}.$$

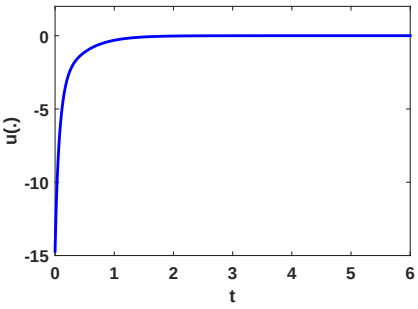
In Figure 1(b) the corresponding stabilizer control is illustrated. Also, in Figures 1(c) and 1(d), the state variables of system (4.1) and its corresponding hyperbolic form are shown, for $\alpha_1 = 0.5$ and $\alpha_2 = -0.5$.

5. Conclusion

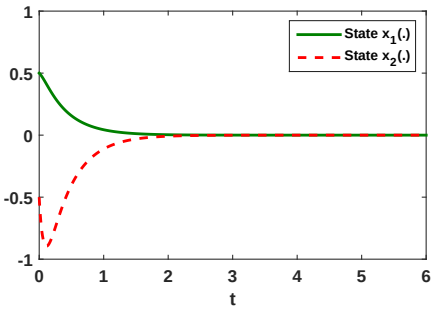
In this paper, we proposed a new approach to design a stabilizer control for nonlinear systems. We showed that the hyperbolic model can be utilized to obtain the stabilizer control for any general nonlinear system. We demonstrated the approach for an inverted pendulum system.



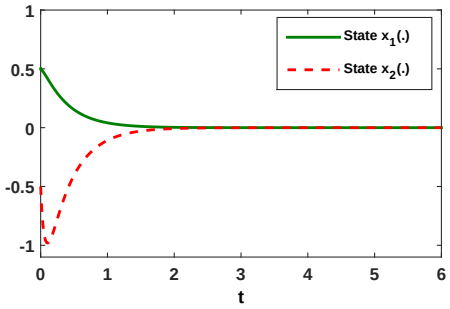
(a) Inverted pendulum system



(b) The stabilizer control



(c) The state for main system



(d) The state for hyperbolic system

Figure 1. Charts

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A NEURO-DYNAMIC SCHEME FOR SOLVING A CLASS OF MINIMAX PROBLEMS

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Abstract. Prior studies indicate that neural network can be used to solve various optimization problems. The main idea of the neural network approach for optimization is to construct a nonnegative energy function and establish a dynamic system that represents an artificial neural network. The dynamic system is usually in the form of first order ordinary differential equations. Furthermore, it is expected that the dynamic system will approach its static state (or an equilibrium point), which corresponds to the solution for the underlying optimization problem, starting from an initial point. In the present paper, a neural network model for solving the quadratic minimax problem is presented. Several numerical simulations are also provided.

1. Introduction

In this paper, we consider the following DQM problem:

$$\min_x \max_y f(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & -Q_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix}, \quad (1.1)$$

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Key words and phrases. Neural network, Minimax problem, Quadratic programming problem, Convergent, Stability.

* Speaker.

subject to

$$d \leq B_1 x + B_2 y \leq h, \tag{1.2}$$

where $Q_{11} \in I\!R^{n\times n}$, $Q_{22} \in I\!R^{m\times m}$ are symmetric and positive semi-definite matrix, $Q_{12} \in I\!R^{n\times m}$, $Q_{21} \in I\!R^{m\times n}$, $B_1 \in I\!R^{l\times n}$ $B_2 \in I\!R^{l\times m}$, $c_1 \in I\!R^n$, $c_2 \in I\!R^m$, and $d, h \in I\!R^l$ and some elements of $-d$ and h can be $+\infty$. In this paper, we denote $\Gamma = \{(x^T, y^T)^T |\ x \in I\!R^n, y \in I\!R^m, d \leq B_1 x + B_2 y \leq h\}$, then a point $(x^{*T}, y^{*T})^T \in \Gamma$ is said to be a saddle point of $f(x, y)$ over the feasible region Γ if

$$f(x^*, y) \leq f(x^*, y^*) \leq f(x, y^*),\ \forall x \in X(y^*)\ , y \in Y(x^*), \tag{1.3}$$

where

$$X(y^*) = \{x \in I\!R^n | d \leq B_1 x + B_2 y^* \leq h\}, \tag{1.4}$$

$$Y(x^*) = \{y \in I\!R^m | d \leq B_1 x^* + B_2 y \leq h\}. \tag{1.5}$$

It is clear that DQM problem (1.1)-(1.2) can be written as follows

$$\min \max\ f(x, y) = x^T Q_{11} x + x^T Q_{12} y + y^T Q_{21} x - y^T Q_{22} y + c_1^T x + c_2^T y \tag{1.6}$$

$$\text{subject to } (x, y) \in \Gamma. \tag{1.7}$$

Let $g_1(x^*, y) = y^T Q_{22} y - ((x^*)^T Q_{12} + (x^*)^T Q_{21}^T + c_2^T) y$ and $g_2(x, y^*) = x^T Q_{11} x + ((y^*)^T Q_{12}^T + (y^*)^T Q_{21} + c_1^T) x$. From Proposition 2.1 in [2, 1], we know that $(x^{*T}, y^{*T})^T$ is a solution of the DQM problem (1.1)-(1.2), if and only if $g_1(x^*, y) + g_2(x, y^*)$ obtains its minimum over Γ at $(x^{*T}, y^{*T})^T$.

Now denote $g(x, y, x^*, y^*) = g_1(x^*, y) + g_2(x, y^*)$; then

$$g(x, y, x^*, y^*) =$$
$$y^T Q_{22} y - ((x^*)^T Q_{12} + (x^*)^T Q_{21}^T + c_2^T) y + x^T Q_{11} x + ((y^*)^T Q_{12}^T + (y^*)^T Q_{21} + c_1^T) x =$$
$$\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} Q_{11} & O_{n\times m} \\ O_{m\times n} & Q_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} Q_{12}(y^*) + Q_{21}^T(y^*) + c_1 \\ -Q_{12}^T(x^*) - Q_{21}(x^*) - c_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where O indicates a zero matrix. For simplicity our discussion, denote

$$Q = \begin{pmatrix} Q_{11} & O_{n\times m} \\ O_{m\times n} & Q_{22} \end{pmatrix},\ G = \begin{pmatrix} Q_{12}(y^*) + Q_{21}^T(y^*) + c_1 \\ -Q_{12}^T(x^*) - Q_{21}(x^*) - c_2 \end{pmatrix},$$
$$w = \begin{pmatrix} x \\ y \end{pmatrix},\ E = (B_1, B_2),$$

then the DQM (1.1)-(1.2) is equivalent to the following problem

$$\min\ g(w) = w^T Q w + G^T w \tag{1.8}$$

subject to

$$d \leq E w \leq h. \tag{1.9}$$

We propose a neural network for solving (1.8)-(1.9) as

$$\frac{dw}{dt} = -\left(2Qw + G + E^T(u_1 + Ew - h)^+ - E^T(u_2 + d - Ew)^+\right), \quad (1.10)$$

$$\frac{du_1}{dt} = (u_1 + Ew - h)^+ - u_1, \quad (1.11)$$

$$\frac{du_2}{dt} = (u_2 + d - Ew)^+ - u_2, \quad (1.12)$$

with the initial point $(w_0^T, u_0^T)^T$. We denote $z = (w^T, u^T)^T \in I\!R^{n+m+2l}$, and

$$A = \begin{pmatrix} E \\ -E \end{pmatrix}, \ b = \begin{pmatrix} h \\ -d \end{pmatrix},$$

and define

$$\eta(z) = \begin{pmatrix} -\left(2Qw + G + A^T(u + Aw - b)^+\right) \\ (u + Aw - b)^+ - u \end{pmatrix}. \quad (1.13)$$

Thus neural network (1.10)-(1.12) can be written as:

$$\frac{dz}{dt} = \theta\eta(z), \quad (1.14)$$

$$z(t_0) = z_0, \ \theta > 0. \quad (1.15)$$

2. Stability and convergence analysis

In this section, we shall study some stability and convergence properties for (1.14)-(1.15).

Theorem 2.1. *Let $z^* = (w^{*T}, u^{*T})^T$ be the equilibrium point of the neural network (1.14)-(1.15). Then z^* is a KKT point of the problem (1.8)-(1.9) and its dual. On the other hand, if $w^* \in I\!R^{n+m}$ is an optimal solution of problem (1.8)-(1.9), then there exists $u^* \in I\!R^{2l}$ such that $z^* = (w^{*T}, u^{*T})^T$ is an equilibrium point of the proposed neural network (1.14)-(1.15).*

Lemma 2.2. *For any initial point $z(t_0) = (w(t_0)^T, u(t_0)^T)^T$, there exists a unique continuous solution $z(t) = (w(t)^T, u(t)^T)^T$ for system (1.14)-(1.15).*

Lemma 2.3. *Let $A \in I\!R^{2l\times(m+n)}$ be of full rank. Then the Jacobian matrix $\nabla\eta(z)$ of the mapping η defined in (1.13) is negative semidefinite matrix.*

Theorem 2.4. *Let the assumption of lemma 2.3 be satisfied. Then the proposed neural network model in (1.14)-(1.15) is globally stable in the Lyapunov sense and is globally convergent to $z^* = (w^{*T}, u^{*T})^T$, where w^* is the optimal solution of (1.8)-(1.9).*

Theorem 2.5. *The convergence rate of the neural network (1.14)-(1.15) increases as θ increases.*

Example 1[3]

$$\min_x \max_y \ f(x,y) = x_1^2 - y_1 - 2y_2 - y_3^2$$

subject to

$$-y_1 - 2y_2 \leq 0.$$

The set consisted of all optimal solutions for this problem is

$$E_N = \{(x_1, y_1, y_2, y_3)^T | x_1 = y_3 = 0, \ y_1 + 2y_2 = 0\}.$$

Figures 1 and 2 display the convergence with $z_{1,0} = (1, -1, 1, -1, 1)^T$ According to the convergence of $x_1(t), y_3(t)$ and $y_1(t) + 2y_2(t)$ we conclude that the output trajectories of the proposed neural network model converges to an element of E_N.

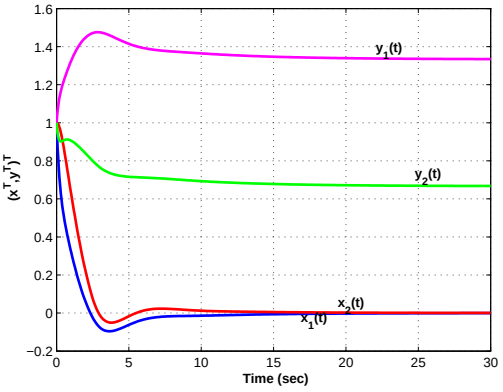

FIGURE 1. The convergence behavior of x_1, x_2, y_1 and y_2 with the initial point $z_{1,0}$ in Example 1.

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OPTIMAL CONTROL OF TEMPERATURE OF A CONTINUOUS-STIRRED TANK REACTOR

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Abstract. In this work, we investigate the optimal control of a continuous-stirred tank reactor. The present problem, by substituting the control variable by its expression derived from the system model in the performance index, is first converted into a variational problem and then its solution is achieved by satisfying Euler-Lagrange equation. Finally, the parametric iteration method is applied to solve the resulting the Euler-Lagrange equation and the optimal control law is readily obtained by simple calculation.

1. Introduction

Here we consider the temperature control problem by cooling-rate manipulation of a continuous-stirred tank reactor (CSTR). The optimal control problem of the temperature is formulated as follows [1]:

$$\min_{u(t)} J = \frac{1}{2}\int_0^{0.5} \left[(x(t)-1.3)^2 + \mu u^2(t)\right] dt, \quad \mu > 0 \tag{1.1}$$

subject to

$$\begin{aligned} &\dot{x}(t) = 1 - x(t) + a\, e^{-\gamma/x(t)} - u(t), \\ &x(0) = 1.5 \quad \text{and} \quad x(0.5) = 1.3. \end{aligned} \tag{1.2}$$

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Key words and phrases. Optimal control, Euler-Lagrange equation, Parametric iteration method.

* Speaker.

The dimensionless variables and the used parameter values in the reactor model (1.2) are as below:

$$\begin{aligned} &x(t)=\tfrac{T}{T_f}, \quad t=\tfrac{\tau}{F/V}, \quad a=-\tfrac{\Delta H k_0 V}{\rho C_p T_f F} \\ &\gamma=\tfrac{E}{RT_f}, \quad u=\tfrac{\dot{Q}}{\rho C_p T_f F}, \quad x_0=\tfrac{T_0}{T_f}, \quad x_d=\tfrac{T_d}{T_f}, \\ &a=1000, \quad \gamma=10, \quad \mu=0.25. \end{aligned}$$

Now one can transform the optimal control (1.1) and (1.2) to a variational problem with the same boundary conditions. To do this, from (1.2), we obtain the expression for $u(t)$ as a function of t, $x(t)$ and $\dot{x}(t)$ as follows:

$$u(t)=1-x(t)+a\,e^{-\gamma/x(t)}-\dot{x}(t), \tag{1.3}$$

which by substituting (1.3) into (1.1) yields the following variational problem

$$\min_{x(t)} J=\frac{1}{2}\int_0^{0.5}\left[(x(t)-1.3)^2+\mu\left(1-x(t)+a\,e^{-\gamma/x(t)}-\dot{x}(t)\right)^2\right]dt, \tag{1.4}$$

and thus

$$G\left(t,x(t),\dot{x}(t)\right)=(x(t)-1.3)^2+\mu\left(1-x(t)+a\,e^{-\gamma/x(t)}-\dot{x}(t)\right)^2. \tag{1.5}$$

Therefore, according to the Euler-Lagrange equation $\frac{\partial G}{\partial x}-\frac{d}{dt}(\frac{\partial G}{\partial \dot{x}})=0$, [2], and by simple operation, we gain the following strongly nonlinear two-point boundary value problem

$$\ddot{x}(t)=\frac{1}{\mu}\left(x(t)-1.3\right)+\left(1-x(t)+a\,e^{-\gamma/x(t)}\right)\left(\frac{a\gamma}{x^2(t)}\,e^{-\gamma/x(t)}-1\right), \tag{1.6}$$

subject to the boundary conditions

$$x(0)=1.5 \quad \text{and} \quad x(0.5)=1.3. \tag{1.7}$$

2. Main results

In general, the Euler-Lagrange equation is nonlinear. So, it is difficult to analytically obtain the solution of the two-point boundary value problem (1.6) with the boundary conditions (1.7). Here we will apply the effective parametric iteration method (PIM) to solve (1.6), which is an approximate analytical method. According to [3], the PIM for solving (1.6) is as follows:

$$x_{n+1}(t)=x_n(t)+h\int_0^t(t-s)F_n(s)\mathrm{d}s-h\frac{t}{0.5}\int_0^{0.5}(0.5-s)F_n(s)\mathrm{d}s, \tag{2.1}$$

with

$$x_{n+1}(0)=1.5 \quad \text{and} \quad x_{n+1}(0.5)=1.3, \tag{2.2}$$

where

$$\mathcal{A}[x_n(s)] = \ddot{x}_n(s) - \frac{1}{\mu}\,(x_n(s) - 1.3) \\ - \big(1 - x_n(s) + a\,e^{-\gamma/x_n(s)}\big)\left(\frac{a\gamma}{x_n^2(s)}\,e^{-\gamma/x_n(s)} - 1\right) \tag{2.3}$$

and

$$\mathcal{A}[x_n(s)] = F_n(s) + O\big((s-\eta)^{n+1}\big). \tag{2.4}$$

with $\eta = (t_0 + t_f)/2$, $t_0 = 0$ and $t_f = 0.5$. This selection of η could uniformly distribute the error across the interval. Now, in light of the PIM procedure (2.1), the boundary conditions (2.2) and by choosing $h = -1$, $\eta = 0.25$ and $x_0(t) = 1.5 - 0.4t$, we will get the following PIM approximations:

$$x_0(t) = 1.5 - 0.4t,$$
$$x_1(t) = 1.5 - 0.7961004060801135043 6t + 0.7922008121602270087 0t^2,$$
$$x_2(t) = 1.5 - 0.7085506513353677101 5t + 1.0470331349744920739t^2 \\ -0.8598636646075133072t^3,$$
$$\dots$$

The absolute errors of the state variable $x(t)$ and the control variable $u(t)$ using the 15th-order PIM approximation are shown in Table 1.

TABLE 1. Absolute errors of $x(t)$ and $u(t)$ using the 15th-order PIM approximation

t	$\lvert u_{15}(t) - u_{Num}(t)\rvert$	$\lvert x_{15}(t) - x_{Num}(t)\rvert$
0.1	1.12×10^{-6}	1.19×10^{-7}
0.2	1.56×10^{-7}	2.17×10^{-7}
0.3	9.17×10^{-7}	2.16×10^{-7}
0.4	1.24×10^{-6}	1.45×10^{-7}

Also, the relative error of the performance index, i.e., $E_n = \lvert\frac{J_{n+1}-J_n}{J_{n+1}}\rvert$ can be seen in Table 2.

TABLE 2. Relative error of the performance index using the 15th-order PIM approximation

$n = 1-5$	$n = 6-10$	$n = 11-15$
$E_1 = 5.4\times10^{-2}$	$E_6 = 2.4\times10^{-5}$	$E_{11} = 1.4\times10^{-8}$
$E_2 = 4.5\times10^{-3}$	$E_7 = 1.1\times10^{-5}$	$E_{12} = 1.4\times10^{-9}$
$E_3 = 5.8\times10^{-3}$	$E_8 = 1.0\times10^{-6}$	$E_{13} = 5.2\times10^{-10}$
$E_4 = 4.4\times10^{-4}$	$E_9 = 4.1\times10^{-7}$	$E_{14} = 5.2\times10^{-11}$
$E_5 = 2.9\times10^{-4}$	$E_{10} = 3.9\times10^{-8}$	$E_{15} = 1.8\times10^{-11}$

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EXISTENCE OF CODIMENSION-ONE BIFURCATIONS IN AN INERTIAL 4-NEURON SYSTEM WITH MULTIPLE DELAYS

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Abstract. In this paper, we present results which are obtained from studying an inertial four-neuron system with multiple delays. This network represents a nonlinear system of ordinary differential equations with different delays. By analyzing its associated characteristic equation, the existence of codimension-one bifurcations of the system is investigated.

1. Introduction

In recent decades, modeling biological neuron has attracted great attention because of studies of Hodgkin and Huxley on the firing activities of squid axon. In 1987, the neuron system was modeled by the circuit with inertial term [1]. Then, Wheeler and Schieve (1997) studied an inertial two-neuron system. Furthermore, time delays often occur during the signal transmission. Marcus and Westervelt presented a neural network with delay, [5]. Studies on neural networks showed that they have complex dynamical behavior, such as chaos and bifurcations. For example, Li et al. [4] considered a single delayed inertial neuron model. They observed Hopf bifurcation and chaotic behavior. After

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* Speaker.

their works, many researchers has been attracted to analyze dynamics of a delayed inertial neural network, [2, 3] and refernces there in. In 2009, Liu et al. discussed on stability of bifurcating periodic solutions for a single delayed inertial neuron model under periodic excitation. Moreover, he studied dynamics of an inertial two-neuron system with time delay in another work. Then, an inertial four-neuron model with time delay was investigated by Ge and Xu (2012 and 2013). In recent years, there are other works on inertial neural systems with time delay such as the study of Song and Xu on an inertial two-neuron coupling system with multiple delays (2014) and the work of Ge and Xu on an inertial four-neuron system with time delay.
Based on the above studies, in this paper, the four-neuron inertial neural system with different time delays is cosidered as follows

$$\begin{aligned}
\ddot{v}_1 &= -\dot{v}_1 - \mu v_1 + af(v_3(t-\tau)) + af(v_4(t-\tau)),\\
\ddot{v}_2 &= -\dot{v}_2 - \mu v_2 + bf(v_3(t-\tau)) + bf(v_4(t-\tau)),\\
\ddot{v}_3 &= -\dot{v}_3 - \mu v_3 + cf(v_1(t-\delta)) + cf(v_2(t-\delta)),\\
\ddot{v}_4 &= -\dot{v}_4 - \mu v_4 + df(v_1(t-\delta)) + df(v_2(t-\delta)).
\end{aligned} \tag{1.1}$$

This system is almost similar to the system of [2]. In this system, a, b, c and d measure the synaptic coupling weights through neurons;$\mu > 0$ describes the stability of internal neuron processes; v_1, v_2, v_3 and v_4 denote the states of the neurons; $\tau, \delta > 0$ represent the time delays in signal transmission between the neurons; $f(.)$ is the nonlinear activation function. In [2], they considered $\tau = \delta$.
In this study, we would like to get the conditions which guarantee the existence of codimension-one bifurcations for this system with different time delays. To the best of our knowledge, we only studied this type of architecture of the four-neuron inertial neural model from the viewpoint of the existence of codimension-one bifurcations.

2. Main results

We consider system 1.1 with the following assumptions:
(H1) $f \in C^1$, $f(0) = 0$, $f'(0) = 1$
(H2) $Sgn((a+b)(c+d)) > 0$,
(H3) $\eta := \delta + \tau$ and $l := (a+b)(c+d)$.
It is obvious that the origin is the equilibrium of 1.1. Let $v_1 = x_1, \dot{v}_1 = x_2, v_2 = x_3, \dot{v}_2 = x_4, v_3 = x_5, \dot{v}_3 = x_6, v_4 = x_7, \dot{v}_4 = x_8$. Then the

following system is topolgically equivalent to 1.1

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -x_2 - \mu x_1 + af(x_5(t-\tau)) + af(x_7(t-\tau)),\\
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= -x_4 - \mu x_3 + bf(x_5(t-\tau)) + bf(x_7(t-\tau)),\\
\dot{x}_5 &= x_6,\\
\dot{x}_6 &= -x_6 - \mu x_5 + cf(x_1(t-\delta)) + cf(x_3(t-\delta)),\\
\dot{x}_7 &= x_7,\\
\dot{x}_8 &= -x_8 - \mu x_7 + df(x_1(t-\delta)) + df(x_3(t-\delta)).
\end{aligned} \tag{2.1}$$

Then, the associated charectristic equation of system 2.1 is as follows

$$P(\lambda,\eta) = p_8(\lambda) + p_4(\lambda)e^{-\eta\lambda}, \tag{2.2}$$

where

$$\begin{aligned}
p_8(\lambda) &= (\lambda^2+\lambda+\mu)^4\\
&= \lambda^8 + 4\lambda^7 + \lambda^6(4\mu+6) + \lambda^5(12\mu+4) + \lambda^4\left(6\mu^2+12\mu+1\right)\\
&+ \lambda^3\left(12\mu^2+4\mu\right) + \lambda^2\left(4\mu^3+6\mu^2\right) + 4\lambda\mu^3 + \mu^4\\
p_4(\lambda) &= -l(\lambda^2+\lambda+\mu)^2\\
&= -l(\lambda^4 + 2\lambda^3 + \lambda^2(2\mu+1) + 2\lambda\mu + \mu^2).
\end{aligned}$$

For simplicity, assume that $q(\lambda) = (\lambda^2+\lambda+\mu)^2$. It is clear that

$$\begin{aligned}
P(\lambda,\eta) &= q^2(\lambda) - lq(\lambda)e^{-\eta\lambda}\\
\frac{dP(\lambda,\eta)}{d\lambda} &= q'(\lambda)(2q(\lambda) - le^{-\eta\lambda}) + l\eta q(\lambda)e^{-\eta\lambda}.
\end{aligned}$$

Now, we can prove that there are several types of bifurcation.

(a) It is not hard to compute

$$\begin{aligned}
P(0,\eta) &= \mu^2(\mu^2 - l)\\
\frac{dP(\lambda,\eta)}{d\lambda}|\lambda = 0 &= l(1+\eta\mu^2).
\end{aligned}$$

If $\mu^2 = l$, then zero is a simple root of 2.2. In this case, codimension-one bifurcations such as saddle-node bifurcation, pitchfork bifurcation or transcritical bifurcation can occur.

(b) Let $\lambda = iw$ $(w > 0)$ be the root of 2.2. Thus we have the following equation by separating the real and imaginary parts

$$\begin{aligned} h(w^2) &= w^{16} + (4-8\mu)w^{14} + \left(28\mu^2 - 24\mu + 6\right) w^{12} \\ &+ \left(-56\mu^3 + 60\mu^2 - 24\mu + 4\right) w^{10} + w^8 \left(70\mu^4 - 80\mu^3 + 36\mu^2 - 8\mu - l + 1\right) \\ &+ w^6 \left(-56\mu^5 + 60\mu^4 - 24\mu^3 + 4\mu^2 + l(4\mu+2) - 4l\right) \\ &+ w^4 \left(28\mu^6 - 24\mu^5 + 6\mu^4 + l\left(-6\mu^2 - 4\mu - 1\right) + 8l\mu\right) \\ &+ w^2 \left(-8\mu^7 + 4\mu^6 - 4l\mu^2 + l\left(4\mu^3 + 2\mu^2\right)\right) + \mu^8 - l\mu^4 \\ &= 0 \end{aligned}$$

If $y = w^2$ and $\mu^4 - l < 0$ then $h(y) = 0$ has a positive root. If w is not a simple root, then $\frac{dP(\lambda,\eta)}{d\lambda}|_{\lambda=iw} = 0$. Hence

$$\begin{aligned} Im(\eta) &= \frac{-i(p_4'(iw)p_4(\bar{iw}) - p_4(iw)p_4'(\bar{iw}))}{2|p_4(iw)|^2} + \frac{i(p_8(iw)p_8'(\bar{iw}) - p_8'(iw)p_8(\bar{iw}))}{2|p_8(iw)|^2} \\ &= 0 \end{aligned}$$

Therefore, If $h'(w^2) \neq 0$ then w is a simple root of 2.2.On the other hand, we have

$$\frac{dRe(\lambda)}{d\eta}|_{\lambda=iw} = \frac{w^2(1-2\mu+2w^2)}{(\eta\mu+1)^2 + (\eta(-2\eta\mu+\eta+2)+4)w^2 + \eta^2 w^4}.$$

By the above discussion, we have the following theorem.

Theorem 2.1. *If one of the following conditions satisfies then one of codimension-one bifurcations such as saddle-node bifurcation, pitchfork bifurcation, transcritical bifurcation or Hopf bifurcation can occur.*

(a) $\mu^2 = l$,

(b) $\mu^4 - l < 0$, $h'(w^2) \neq 0$ *and* $(1 - 2\mu + 2w^2) \neq 0$.

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SOLVING OPTIMAL CONTROL PROBLEM WITH CHEBYSHEV POLYNOMIALS

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Abstract. We apply the variational iteration method (VIM) to approximate the solution of a class of optimal control problems (OCPs). First, we propose an approximated function based on Chebyshev polynomials for the control function in the OCP. Then this problem is converted to a problem in the calculus of variations. By using the VIM, we obtain the approximated solution for the state function of the OCP. Then by using a optimization approach for cost functional, we determine unknown coefficients for control function.

1. Introduction

Chebyshev polynomials $T_n(z)$ are defined as:

$$T_n(z) = \cos(n \cos^{-1}(z)); \quad -1 \leq z \leq +1$$

where the independent variable z is defined between -1 and $+1$. Chebyshev polynomials can be obtained by means of the following recurrence formula:

$$\begin{aligned} T_0(z) &= 1, \\ T_1(z) &= z, \end{aligned}$$

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* Speaker.

$$T_{n+1}(z) \quad = \quad 2zT_n(z) - T_{n-1}(z), \quad n = 1, 2, ...$$

then an arbitrary function $f(z)$ can be approximated by the Chebyshev polynomials:

$$f(z) = \sum_{n=0}^{N} b_n T_n(z),$$

where b_n are unknown coefficients.

Now we describe the OCP. It is the problem of finding an optimal control $u(t)$ that minimizes the cost functional:

$$J(u(t)) = \int_a^b f(x(t), u(t), t)\ dt, \tag{1.1}$$

which satisfies the following dynamical system:

$$\dot{x}(t) = g(x(t), u(t), t), \tag{1.2}$$

and the initial condition:

$$x(a) = x_a. \tag{1.3}$$

where $x(t)$, denotes the state variable, $u(t)$ is the control variable, f and g are two continuously differentiable functions with respect to all their arguments.

2. Main results

In this section we explain the VIM for solving the above OCP. To illustrate its basic idea of the technique, we consider the following general nonlinear system:

$$L[x(t)] + N[x(t)] = g(t), \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a given continuous function. The basic character of the method is to construct a correction functional for system (2.1) which reads

$$x_{n+1}(t) = x_n(t) + \int_0^t \lambda(s)[Lx_n(s) + N\tilde{x}_n(s) - g(s)]\ ds \tag{2.2}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory, $x_n(t)$ is the nth approximate solution, and $\tilde{x}_n$ denotes a restricted variation, i.e. $\delta\tilde{x}_n = 0$.

It is obvious that in the VIM, we first require the determination of the Lagrangian multiplier λ. Having determined the Lagrangian multiplier, the successive approximations $x_n(t), n \geq 0$, of the solution $x(t)$ will be readily obtained by choosing $x_0(t)$. Consequently, the solution will be as:

$$x(t) = \lim_{n\to\infty} x_n(t). \tag{2.3}$$

Now to solve the OCP (1.1)-(1.3), we first propose an approximated function for $u(t)$ based on Chebyshev polynomials like $u_N(t) = \sum_{k=0}^{N} b_k T_k(t)$. Then we substitute the approximated control function in cost functional and ordinary dynamical system, therefore we obtain the calculus of variations problem. Now we use the VIM for solving the resulted differential equation as below:

$$\begin{aligned} x_{n+1}(t) &= x_n(t) + \int_a^t \lambda(s) g(x_n(s), \sum_{k=0}^{N} b_k T_k(s), s)\ ds, \\ x_0(t) &= x(a) = x_a. \end{aligned} \tag{2.4}$$

Example 2.1. Consider the following OCP, to minimize

$$J(u(t)) = \frac{1}{2}\int_0^1 [x_1(t)^2 + x_2(t)^2 + u(t)^2]\ dt, \tag{2.5}$$

subject to the dynamic constraints:

$$\dot{x}_1(t) = -x_1(t) + x_2(t) + u(t), \tag{2.6}$$

$$\dot{x}_2(t) = -2x_2(t), \tag{2.7}$$

and the initial conditions:

$$x_1(0) = x_2(0) = 1.$$

The exact solution for this OCP is:

$$\begin{aligned} x_1(t) &= -\frac{3}{2}e^{-2t} + 2.48164e^{-\sqrt{2}t} + 0.018352e^{\sqrt{2}t} \\ x_2(t) &= e^{-2t} \\ u(t) &= \frac{1}{2}e^{-2t} - 1.02793e^{-\sqrt{2}t} + 0.0443056e^{\sqrt{2}t} \end{aligned}$$

The optimal value of performance index for this problem is $J^* = 0.4319835549$.

Now we suppose that $u_N(t) = \sum_{k=0}^{N} b_k T_k(t)$, where $T_k(t)$ is kth Chebyshev polynomial.

By using formula (2.4), we have the following recurrence relations:

$$\begin{aligned} x^1_{n+1}(t) &= x^1_n(t) - \int_0^t [\dot{x}^1_n(s) + x^1_n(s) - x^2_n(s) - \sum_{k=0}^{N} b_k T_k(s)]\ ds \\ x^1_0(t) &= 1 \\ x^2_{n+1}(t) &= x^2_n(t) - \int_0^t [\dot{x}^2_n(s) + 2x^2_n(s)]\ ds \end{aligned}$$

$$x_0^2(t) = 1 \tag{2.8}$$

The numerical results can be observed in Table 1.

TABLE 1. Different optimal values of cost functional J

N	n	Different optimal values of cost functional J
3	10	0.431987476
3	12	0.431987281
3	15	0.431987278
4	10	0.431987439
4	12	0.431987244
4	15	0.431987240

Table 1 shows that the developed technique is effective and accurate for the OCPs.

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A NEW SOLUTION OF DELAY SYSTEMS

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Abstract. The Bezier curve method is presented to solve delay systems. A direct algorithm for solving this problem is given. The delay function and inverse time function are expanded by Bezier curves. We have chosen the Bezier curves as piecewise polynomials of degree n, and determine Bezier curves on any subinterval by $n+1$ control points. The approximate solution of delay systems containing inverse time is derived.

1. Introduction

This paper goals at solving delay systems containing inverse time of the following form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + C(t)(x_1(t-\tau_1)\dots x_p(t-\tau_p))^T \\ &+ D(t)(x_1(t_f-t)\dots x_p(t_f-t))^T + G(t)\mathbf{u}(t), \\ \mathbf{x}(t) &= \boldsymbol{\phi}(t),\ t\in[-\tau_{\max}, t_0], \end{aligned} \tag{1.1}$$

where $\mathbf{x}(t) = (x_1(t)\dots x_p(t))^T \in \mathbb{R}^p$, $\mathbf{u}(t) = (u_1(t)\dots u_m(t))^T \in \mathbb{R}^m$ are respectively state and control functions while $\boldsymbol{\phi}(t) = (\phi_1(t)\dots\phi_p(t))^T$, is known vector function and τ_i's $(i = 1, 2, \dots, p)$ are non-negative constant time delays, and $\tau_{max} = \max\{\tau_i,\ 1 \le i \le p\}$. We assume the

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matrices $A(t) = [a_{ij}(t)]_{p\times p}$, $C(t) = [c_{ij}(t)]_{p\times p}$, $D(t) = [d_{ij}(t)]_{p\times p}$ and $G(t) = [g_{ij}(t)]_{m\times m}$ are matrix functions.

2. Function approximation

Divide the interval $[t_0, t_f]$ into a set of grid points such that

$$t_i = t_0 + ih,\ i = 0,\ 1,\ \ldots,\ k,$$

where $h = \frac{t_f - t_0}{k}$, and k is a positive integer. Let $S_j = [t_{j-1}, t_j]$ for $j = 1,\ 2,\ \ldots,\ k$. Then, for $t \in S_j$, delay systems containing inverse time (1.1) can be decomposed to the following problems:

$$\begin{aligned}
\dot{\mathbf{x}}_j(t) &= A(t)\mathbf{x}_j(t) + C(t)(x_1^{-k_1^1+j}(t-\tau_1)\ldots x_p^{-k_1^p+j}(t-\tau_p))^T \\
&+ D(t)(x_1^{k_2^1-j}(t_f-t)\ldots x_p^{k_2^p-j}(t_f-t))^T + G(t)\mathbf{u}_j(t), \\
\mathbf{x}(\theta) &= \boldsymbol{\phi}(\theta),\ \theta \in [-\tau_{max}, t_0],
\end{aligned} \tag{2.1}$$

where $\mathbf{x}_j(t) = (x_1^j(t)\ldots x_p^j(t))^T$, and $\mathbf{u}_j(t) = (u_1^j(t)\ldots\ u_m^j(t))^T$ are respectively vectors of $\mathbf{x}(t)$ and $\mathbf{u}(t)$ which are considered in $t \in S_j$, we mention that $x_i^{-k_1^i+j}(t-\tau_i)$; $1 \le i \le p$, is the i-th component of $(x_1^{-k_1^1+j}(t-\tau_1)\ldots x_p^{-k_1^p+j}(t-\tau_p))^T$ where $(t-\tau_i) \in [t_{-k_1^i+j-1}, t_{-k_1^i+j}]$, and $x_i^{-k_2^i+j}(t_f-t)$; $1 \le i \le p$ is the i-th component of $(x_1^{k_2^1-j}(t_f - t)\ldots x_p^{k_2^p-j}(t_f-t))^T$ where $(t_f-t) \in [t_{k_2^i-j-1}, t_{k_1^i-j}]$, Also

$$k_1^i = \begin{cases} \frac{\tau_i}{h} & \frac{\tau_i}{h} \in \mathbb{N} \\ ([\frac{\tau_i}{h}]+1) & \frac{\tau_i}{h} \notin \mathbb{N} \end{cases},\ 1 \le i \le p,$$

$$k_2^i = \begin{cases} \frac{t_f}{h} & \frac{t_f}{h} \in \mathbb{N} \\ ([\frac{t_f}{h}]+1) & \frac{t_f}{h} \notin \mathbb{N} \end{cases}\ 1 \le i \le p,$$

where $[\frac{\tau_i}{h}]$ and $[\frac{t_f}{h}]$ denote the integer part of $\frac{\tau_i}{h}$ and $\frac{t_f}{h}$ respectively.
Our strategy is using Bezier curves to approximate the solutions $\mathbf{x}_j(t)$ and $\mathbf{u}_j(t)$ by $\mathbf{v}_j(t)$ and $\mathbf{w}_j(t)$ respectively, where $\mathbf{v}_j(t)$ and $\mathbf{w}_j(t)$ are given below. Individual Bezier curves that are defined over the subintervals are joined together to form the Bezier spline curves. For $j = 1, 2, \ldots, k$, define the Bezier polynomials of degree n that approximate respectively the actions of $\mathbf{x}_j(t)$ and $\mathbf{u}_j(t)$ over the interval $[t_{j-1}, t_j]$ as follows

$$\begin{aligned}
\mathbf{v}_j(t) &= \sum_{r=0}^{n} \mathbf{a}_r^j B_{r,n}(\frac{t-t_{j-1}}{h}), \\
\mathbf{w}_j(t) &= \sum_{r=0}^{n} \mathbf{b}_r^j B_{r,n}(\frac{t-t_{j-1}}{h}),
\end{aligned} \tag{2.2}$$

where

$$B_{r,n}(\frac{t-t_{j-1}}{h}) = \binom{n}{r}\frac{1}{h^n}(t_j-t)^{n-r}(t-t_{j-1})^r,$$

is the Bernstein polynomial of degree n over the interval $[t_{j-1}, t_j]$, $\mathbf{a}_r^j$ and $\mathbf{b}_r^j$ are respectively p and m ordered vectors from the control points. Now, we have

$$\begin{aligned} R_{1,j}(t) &= \dot{\mathbf{v}}_j(t) - A(t)\mathbf{v}_j(t) - C(t) \\ &\quad (v_1^{-k_1^1+j}(t-\tau_1)\dots v_p^{-k_1^p+j}(t-\tau_p))^T \\ &- D(t)(v_1^{k_2^1-j}(t_f-t)\dots v_p^{k_2^p-j}(t_f-t))^T - G(t)\mathbf{w}_j(t), \end{aligned} \tag{2.3}$$

Now, we define the residual function in S_j as follows

$$R_j = \int_{t_{j-1}}^{t_j} (M\|R_{1,j}(t)\|^2)dt, \tag{2.4}$$

where $\|.\|$ is the Euclidian norm (Recall that $R_{1,j}(t)$ is a p vector where $t \in S_j$) and M is a sufficiently large penalty parameter. Our aim is to solve the following problem over $S = \bigcup_{j=1}^k S_j$:

$$\begin{aligned} \min \quad & \sum_{j=1}^{k} R_j \\ s.t. \quad & \mathbf{a}_n^j(t_j-t_{j-1})^n = \mathbf{a}_0^{j+1}(t_{j+1}-t_j)^n, \\ & (\mathbf{a}_n^j - \mathbf{a}_{n-1}^j)(t_j-t_{j-1})^{n-1} = (\mathbf{a}_1^{j+1} - \mathbf{a}_0^{j+1})(t_{j+1}-t_j)^{n-1} \\ & , \ j = 1, 2, \dots, k-1. \end{aligned} \tag{2.5}$$

The mathematical programming problem (2.5) can be solved by many subroutine algorithms, we used Maple 12 to solve this optimization problem.

3. NUMERICAL EXAMPLES

Example 3.1. Consider the delay system containing inverse time described by (see [1]),

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} t^2+1 & -t^2 \\ 0 & -9 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 & -1 \\ 9 & 0 \end{bmatrix}\mathbf{x}(t-\frac{1}{3}) \\ &+ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\mathbf{x}(1-t) + \begin{bmatrix} 4t+3 \\ 8t+15 \end{bmatrix}u(t), \\ \boldsymbol{\phi}(t) &= \begin{bmatrix} t^2-1 \\ t^2+1 \end{bmatrix}, \ t \in [-\frac{1}{3}, 0] \end{aligned}$$

where we have the following exact solution

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T = \begin{bmatrix} t^2 - 1 & t^2 + 1 \end{bmatrix}^T$$

Now, by Equation (2.5) and choosing $n = 3$, $k = 6$ we have the approximated solution $\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$

$$x_1(t) = \begin{cases} -1.000000001 + 8.333333337 * 10^{-9}t + .9999999669t^2 + 10^{-7}t^3, & 0 \le t \le \frac{1}{6}, \\ -.9999999988 + 8.13333333 * 10^{-9}t + .9999999829t^2, & \frac{1}{6} \le t \le \frac{1}{3}, \\ -.9999999997 + 2.00 * 10^{-10}t + t^2, & \frac{1}{3} \le t \le \frac{1}{2}, \\ -.9999999927 - 2.202222223 * 10^{-8}t + 1.000000017t^2, & \frac{1}{2} \le t \le \frac{2}{3}, \\ -.9999999902 - 1.504444443 * 10^{-8}t + .9999999963 * t^2 + 10^(-8) * t^3, & \frac{2}{3} \le t \le \frac{5}{6}, \\ -1.000000032 + 1.120666667 * 10^{-7}t + .9999998702t^2 + 5 * 10^{-8}t^3, & \frac{5}{6} \le t \le 1, \end{cases}$$

$$x_2(t) = \begin{cases} 1.000000001 + 0.000011825t + .9996447669t^2 + 0.0023693t^3, & 0 \le t \le \frac{1}{6}, \\ 1.000000001 + 0.00001180813339t + .9996447663t^2 + 0.0023695t^3, & \frac{1}{6} \le t \le \frac{1}{3}, \\ .9999999645 + 0.00001211131104t + .9996439669t^2 + 0.0023702t^3, & \frac{1}{3} \le t \le \frac{1}{2}, \\ 1.000000063 + 0.00001151408882t + .9996452169t^2 + 0.0023693t^3, & \frac{1}{2} \le t \le \frac{2}{3}, \\ .9581187057 + .1594325022t + .8040813829t^2 + 0.0783674t^3, & \frac{2}{3} \le t \le \frac{5}{6}, \\ .9581181451 + .1594344559t + .8040791002t^2 + 0.0783683t^3, & \frac{5}{6} \le t \le 1. \end{cases}$$

4. Conclusions

Using the Bezier curves, we provide the general algorithm for the delay systems containing inverse time function and reduce it into a set of algebraic equations. It is also shown that the results can be applied to the boundary value problem. Numerical example shows that the proposed method is efficient and very easy to use.

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A MODIFICATION OF LEGENDRE-GAUSS COLLOCATION METHOD FOR SOLVING A CLASS OF FRACTIONAL OPTIMAL CONTROL PROBLEMS

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Abstract. The optimal conditions for the fractional optimal control problems (FOCPs) are derived in which the fractional differential operators defined in terms of Caputo sense reduce this problem to a system of fractional differential equations (FDEs) that is called two-point boundary value (TPBV) problem. An approximate solution of this problem is constructed by using the Legendre-Gauss collocation method such that the exact boundary conditions are satisfied. Several examples are given and the optimal errors are obtained for the sake of comparison. The results show that the technique introduced here is accurate and easily applied to solve the FOCPs.

1. Introduction

Here, we would like to investigate the possibility of presence numerical approximated solutions for a class of FOCPs. To proceed, we achieved the necessary conditions of optimization for this class of FOCPs with a system of FDEs. To solve this system, first using a modified approach Caputo fractional derivatives (CFD) that our problem relies on. By using this approach and a joint application of Legendre

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* Speaker.

polynomials, we transform the original system of FDEs into a discrete system of ordinary differential equations, in way by obtaining the optimal solutions of this system, we obtain the approximate solution of the FOCP.

1.1. **Fractional Calculus.**

Definition 1.1. The left and right CFDs of order $\alpha \in \mathbb{R}_+$ are defined respectively, by ${}^{C}_{t_0}D^{\alpha}_{t} f(x) := I^{n-\alpha}_{t_0}\, {}_{t_0}D^{n}_{t} f(x)$ and ${}^{C}_{t}D^{\alpha}_{t_f} f(x) := (-1)^n I^{n-\alpha}_{t_f}\, {}_{t}D^{n}_{t_f} f(x)$ with $n = [\alpha] + 1$; that is

$$ {}^{C}_{t_0}D^{\alpha}_{t} f(x) := \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{x} \frac{f^{(n)}(t)}{(x-t)^{n-\alpha-1}} dt \tag{1.1} $$

and

$$ {}^{C}_{t}D^{\alpha}_{t_f} f(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{x}^{t_f} \frac{f^{(n)}(t)}{(t-x)^{n-\alpha-1}} dt \tag{1.2} $$

where $t_0 \le x \le t_f$ and $f^{(n)}(t) = \frac{d^n f(t)}{dt^n} \in L_1[t_0, t_f]$ is the ordinary derivative of integer order n.

1.2. **Legendre-Gauss Collocation Method.**

Any function $f(t) \in L_2[t_0, t_f]$ can be approximated as follows [1]:

$$ f(t) \sim \sum_{n=0}^{\infty} \hat{c}_n \hat{P}_n(t); \quad \hat{c}_n = \frac{2n+1}{t_f - t_0} \int_{t_0}^{t_f} f(t) \hat{P}_n(t) dt, \quad t \in [t_0, t_f], \tag{1.3} $$

where $\hat{P}_n(t) = P_n\Big(\dfrac{2t}{t_f - t_0} - \dfrac{t_f + t_0}{t_f - t_0}\Big)$ are the modified Legendre polynomials (MLPs) of degree at most n. Therefore, if we have:

$$ \begin{cases} \dfrac{d}{dt} u(t) = f(u(t), t), \quad t_0 \le t \le t_f \\ u(t_0) = u_0, \end{cases} \tag{1.4} $$

the Legendre-Gauss collocation method for solving this problem is equivalent to solve the following problem:

$$ \begin{cases} \dfrac{d}{dt} u^M(\hat{t}^M_j) = f(u^M(\hat{t}^M_j), \hat{t}^M_j), \\ u^M(t_0) = u_0, \quad 1 \le j \le M, \end{cases} \tag{1.5} $$

where $u^M(t) = \sum_{n=0}^{M} \hat{c}_n \hat{P}_n(t)$ and $\hat{P}_M(t_0, t_f)$ is the set of MLPs of degree at most M and $\hat{t}^M_j$, $1 \le j \le M$, are the nodes of the MLPs interpolation on $[t_0, t_f]$. To get the answer of equation (1.4), it's enough to obtain coefficients $\hat{c}_n$.

2. Numerical Scheme for Solving FOCPs

Consider:

$$\min J(u) = \frac{1}{2}\int_{t_0}^{t_f} \Big\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\Big\}dt \tag{2.1}$$

$$\begin{aligned} s.t.\ \ {}_{t_0}^{C}D_t^{\alpha}x(t) &= A(t)x(t) + B(t)u(t),\\ x(t_0) &= x_0,\ 0 \le \alpha \le 1 \end{aligned}$$

The aim is to find a control vector $u^*(t)$ such that the cost functional (2.1) is minimized while the dynamic equality constraint is satisfied. So, we define $H(x(t), u(t), \lambda(t), t) = \frac{1}{2}\Big\{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\Big\} + \lambda^T\Big\{A(t)x(t) + B(t)u(t)\Big\}$ where $\lambda \in \mathbb{R}^n$ is the vector of the Lagrange multiplier. Then, we obtain the following TPBVP (see [2]):

$$ {}_{t}^{C}D_{t_f}^{\alpha}\lambda(t) = \frac{\partial H}{\partial x} = Q(t)x(t) + A^T(t)\lambda(t),\ \lambda(t_f) = 0 \tag{2.2}$$

$$\frac{\partial H}{\partial u} = R(t)u(t) + B^T(t)\lambda(t) = 0$$

$$t_0{}^{C}D_t^{\alpha}x(t) = \frac{\partial H}{\partial \lambda} = A(t)x(t) + B(t)u(t),\ x(t_0) = x_0.$$

Now, we use approximation [3]:

$$\begin{cases} A(\alpha,N)(t_f-t)^{-\alpha}\lambda(t) - B(\alpha,N)(t_f-t)^{1-\alpha}\dot{\lambda}(t) + \sum_{p=2}^{N} C(\alpha,p)(t_f-t)^{1-p-\alpha}W_p(t)\\ \qquad -\dfrac{\lambda(t_f)(t_f-t)^{-\alpha}}{\Gamma(1-\alpha)} = Q(t)x(t) + A^T(t)\lambda(t),\\ A(\alpha,N)(t-t_0)^{-\alpha}x(t) + B(\alpha,N)(t-t_0)^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{N} C(\alpha,p)(t-t_0)^{1-p-\alpha}V_p(t)\\ \qquad -\dfrac{x(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} = A(t)x(t) - B(t)R^{-1}(t)B^T(t)\lambda(t),\\ \dot{V}_p(t) = (1-p)(t-t_0)^{p-2}x(t), \quad V_p(t_0) = 0, \qquad p = 2,3,\cdots,N,\\ \dot{W}_p(t) = -(1-p)(t_f-t)^{p-2}\lambda(t), \quad W_p(t_f) = 0, \qquad p = 2,3,\cdots,N,\\ \qquad x(t_0) = x_0, \quad \lambda(t_f) = 0. \end{cases} \tag{2.3}$$

by assuming that $\lambda^M(t) = \sum_{n=0}^{M} \hat{a}_n\hat{P}_n(t)$, $x^M(t) = \sum_{n=0}^{M} \hat{b}_n\hat{P}_n(t)$, $V_p^M(t) = \sum_{n=0}^{M} \hat{c}_n\hat{P}_n(t)$, $W_p^M(t) = \sum_{n=0}^{M} \hat{d}_n\hat{P}_n(t)$.

3. Numerical Examples

Consider the following FOCP:

$$\min J(u) = \frac{1}{2}\int_0^1 \Big\{(x(t)-t^2)^2 + (u(t)-t+1)^2\Big\}dt \tag{3.1}$$

$$s.t.\ {}_0^C D_t^\alpha x(t) = \frac{\Gamma(3)}{\Gamma(2)}(x(t)-tu(t)), \qquad x(0)=0.$$

The exact solution of this equation is given by $x(t) = t^2$, $u(t) = t-1$ when $\alpha = 1$. Table 1 shows the maximum absolute errors of this approximation for $x(t)$ and $u(t)$.

Table 1. Absolute errors of $x(t)$ and $u(t)$ at $\alpha = 1$.

t	**x(t)**	**u(t)**
0.0	$0.50E-10$	0
0.2	$0.52E-10$	$0.3E-12$
0.4	$0.61E-10$	$0.1E-12$
0.6	$0.12E-11$	$0.1E-12$
0.8	$0.20E-11$	$0.1E-12$
1.0	$0.35E-11$	$0.1E-12$

4. Conclusions

We developed a new approach for solving a class of FOCPs that is based on the Legendre-Gauss collocation method. Numerical results show that this approximation is computationally attractive and also reduces keeping the accuracy of the solution.

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FRAME DUALITY PROPERTIES FOR PROJECTIVE UNITARY REPRESENTATIONS

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Abstract. Let π be the projective unitary representation of a discrete countable Abelian group on a separable Hilbert space. If the set B_π of Bessel vectors for π is dense in $\mathcal{H}$, then for any vector $x \in \mathcal{H}$ the analysis operator θ_x make sense as a densely defined operator from B_π to $l^2(G)$-space. Morever, we investigate the concepts π-orthogonal and π-weakly equivalent. In addition, two vectors x and y are called π-orthogonal if the range spaces of θ_x and θ_y are orthogonal, and they are π-weakly equivalent if the closure of the ranges of θ_x and θ_x are the same.

1. Introduction

We will present some results on a duality property for orthogonal (that is, strongly disjoint) and weakly equivalent frame-generator vectors for group representations and; more generally, projective unitary representations. Our main results are Theorems 2.5 and 2.7. Our main focus on this paper is to investigate the duality connections for general unitary systems associated with projective unitary representations of countable groups. We subsequently discovered that we were really proving results for projective group representations. In this article we

Key words and phrases. Projective unitary representations, π-orthogonal, π-weakly equivalent.

* Speaker.

will use the projective terminology in discussing our earlier results from [1],[3] and [4]. The set of group representations of a group G is a subset of the set of projective group representations. Moreover, most proofs concerning frames for projective group representations are not much more difficult. In most cases we will only state and prove the projective unitary representation case. Here we state some prerequisite that will be needed in the sequel.

Definition 1.1. A frame for a Hilbert space $\mathcal{H}$ is a sequence $\{x_n\}$ in $\mathcal{H}$ with the property that there exist positive constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{g\in G} |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \tag{1.1}$$

holds for every $x \in \mathcal{H}$. A tight frame refers to the case when A = B, and a Parseval frame refers to the case when A = B = 1. In the case that (1.1) holds only for all $x \in \overline{\text{span}}\{x_n\}$, then we say that $\{x_n\}$ is a frame sequence. If we require only the right-hand side of the inequality (1.1), then $\{x_n\}$ is called a Bessel sequence.

In following we investigate the concepts of wandering frame collection and complete wandering collection.

Definition 1.2. A collection of vectors $W = \{w_1, \cdots, w_n\}$ will be called a wandering frame collection (or complete wandering frame collection) for π if the collection $S = \{\pi_g w_i : g \in G, i = 1, \cdots, n\}$ is a frame for its closed linear span (or for $\mathcal{H}$).
A vector $\xi \in \mathcal{H}$ is called a complete frame vector, a complete tight frame vector, or a complete Parseval frame vector for π if $\{\pi(g)\xi : g \in G\}$ is a frame, tight frame or Parseval frame, respectively for the whole Hibert space $\mathcal{H}$.

Let M be a subset of a Hilbert space $\mathcal{H}$ and let $\mathcal{A}$ be a subset of the space $B(\mathcal{H})$ of all the bounded linear operators on $\mathcal{H}$. In what follows we will use $[M]$ to denote the closed linear span of M. Also $\mathcal{A}'$ denotes the commutant $\{T \in B(\mathcal{H}) : TA = AT, \forall A \in \mathcal{A}\}$ of $\mathcal{A}$.

2. Main results

In this section, we introduce the concept of projective unitary representation which is the extension of unitary representation.

Definition 2.1. A projective unitary representation π for a countable group G is a mapping $g \longrightarrow \pi(g)$ from G into the group $U(\mathcal{H})$ of all the unitary operators on a separable Hilbert space $\mathcal{H}$ such that

$\pi(g)\pi(h) = \mu(g,h)\pi(gh)$ for all $g, h \in G$, where $\mu(g,h)$ is a scalar-valued function on $G \times G$ taking values in the circle group T. This function $\mu(g,h)$ is then called a multiplier of π. In this case, we also say that π is a μ-projective unitary representation.

In the following lemma we establish some properties of $\mu(.,.)$.

Lemma 2.2. *The followings hold*
(i) $\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mu(g_1g_2, g_3)\mu(g_1, g_2)$ *for all* $g_1, g_2, g_3 \in G$.
(ii) $\mu(g, e) = \mu(e, g)$ *for all* $g \in G$, *where* e *denotes the identity of* G.
Any function $\mu : G \times G \longrightarrow T$ *satisfying in (i) and (ii) will be called a multiplier or 2-cocycle of* G [2]. *Also, from (i) and (ii) we have*
(iii) $\mu(g, g^{-1}) = \mu(g^{-1}, g)$ *for all* $g \in G$.

Proof. (i) Consider $\pi(g_1), \pi(g_2), \pi(g_3)$. We compute $\mu(g_1, g_2g_3)\mu(g_2, g_3) = \mu(g_1g_2, g_3)\mu(g_1, g_2)$ for all $g_1, g_2, g_3 \in G$ in two ways,

$$\begin{aligned}
(\pi(g_1)\pi(g_2))\pi(g_3) &= \mu(g_1, g_2)\pi(g_1g_2)\pi(g_3) \\
&= \mu(g_1, g_2)\mu(g_1g_2, g_3)\pi(g_1g_2g_3) \\
\pi(g_1)(\pi(g_2)\pi(g_3)) &= \pi(g_1)\mu(g_2, g_3)\pi(g_2g_3) \\
&= \mu(g_2, g_3)\mu(g_1, g_2g_3)\pi(g_1g_2g_3).
\end{aligned}$$

Comparing these two relations, we have done.
(ii) Similarly, $\pi(g)\pi(e) = \mu(g, e)\pi(ge)$.
$\pi(e)\pi(g) = \mu(e, g)\pi(eg)$.
(iii) By considering $g_3 := g_2^{-1}$ in (i), $\mu(g_1, g_2g_2^{-1})\mu(g_2, g_2^{-1}) = \mu(g_1g_2, g_2^{-1})\mu(g_1, g_2)$.
Again we consider $g_1 := g_2^{-1}$, $\mu(g_2, g_2^{-1}) = \mu(g_2^{-1}g_2, g_2^{-1})\mu(g_2^{-1}, g_2)$. □

For any projective unitary representation π of a countable group G on a Hilbert space H and $x \in \mathcal{H}$, the analysis operator θ_x for x from $D(\theta_x)(\subseteq \mathcal{H})$ to $l^2(G)$ is defined by $\theta_x(y) = \sum_{g\in G}\langle y, \pi(g)x\rangle\chi_g$, where $D(\theta_x) = \{y \in \mathcal{H} : \sum_{g\in G} |\langle y, \pi(g)x\rangle|^2 < \infty\}$ is the domain space of θ_x. Clearly, $B_\pi \subseteq D(\theta_x)$ holds for every $x \in \mathcal{H}$. In the case that B_π is dense in $\mathcal{H}$, we have θ_x as a densely defined and closable linear operator from B_π to $l^2(G)$. Next we investigate the concepts of π-orthogonal and π-weakly equivalent.

Definition 2.3. We will say that two vectors x and y in $\mathcal{H}$ are π-orthogonal if the ranges of θ_x and θ_y are orthogonal. The π-orthogonality definition can be extended in an obvious way to a set of several (or even infinitely many) vectors.
Two vectors x and y are π-weakly equivalent if the closures of the ranges of θ_x and θ_y are the same.

Lemma 2.4. *The operators λ_g and r_g are unitary operators on $l^2(G)$. Also, we have: $\lambda_g\chi_h = \mu(g,h)\chi_{gh}, \quad h \in G$, and $\quad r_g\chi_h = \mu(h,g^{-1})\chi_{hg^{-1}}$, $h \in G$. The following theorem characterizes the π-orthogonality and π-weak equivalence in terms of the commutant of $\pi(G)$.*

Theorem 2.5. *Let π be a projective unitary representation of a countable group G on a Hilbert space $\mathcal{H}$ such that B_π is dense in $\mathcal{H}$, and let $x, y \in \mathcal{H}$. Then,*
(i)Vectors x and y are π-orthogonal, if and only if, $[\pi(G)'x] \perp [\pi(G)'y]$ (or equivalently, $x \perp [\pi(G)'y]$).
(ii) Vectors x and y are π-weakly equivalent, if and only if, $[\pi(G)'x] = [\pi(G)'y]$. There is a 1-1 correspondence between the range of θ_x and $[\pi(G)'x]$. In the rest of the paper we investigate the concept of orthogonality index of π.

Definition 2.6. For a projective unitary representation π of a countable group G on a Hilbert space $\mathcal{H}$, we define the decomposition space of π to be the subspace $D_\pi = \overline{\text{span}}\{\theta_\xi(\mathcal{H}) : \xi \in B_\pi\}$ of $l^2(G)$. We call N the orthogonality index of π if N is the smallest natural number such that there exist N strongly disjoint vectors $\xi_i \in B_\pi$ such that $\{\theta_{\xi_i}(B_\pi) : i = 1, \cdots, N\}$ generates D_π. We say that π has the orthogonality index ∞ if no such a finite integer N exists. The cyclic multiplicity of $\pi(G)'$ is the smallest natural number K such that $\overline{\text{span}}\{\pi(G)'x_i : 1 \leq i \leq K\} = \mathcal{H}$ and $[\pi(G)'x_i] \perp [\pi(G)'y_j]$ when $i \neq j$.

The following theorem characterizes the orthogonality index of π by using the commutant of $\pi(G)$.

Theorem 2.7. [4] *Let π be a projective unitary representation of a countable group G on a Hilbert space $\mathcal{H}$ such that B_π is dense in $\mathcal{H}$. Then, π has the orthogonality index N, if and only if, $\pi(G)'$ has the cyclic multiplicity N, where $\pi(G)'$ denotes the commutant of $\pi(G)$.*

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Posters

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DRUG THERAPY ON INTERACTION BETWEEN TUMOR AND IMMUNE CELLS BY OPTIMAL FRACTIONAL CONTROL THEORY

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Abstract. In this article, we study the dynamic behavior describing the transaction between bodies effective T-cell, naive T-cell and chronic myelogenous leukemia in one side and drug in the other side. The most important feature of the equations with fractional order derivatives is their non-localization. Using this system, we will study the optimized drug dose in chronic myelogenous leukemia treatment with two methods namely targeted therapy and broad cytotoxic therapy. Even the drug dose is important for cancer specialists, the weakness of immunology system in cancer affected patients, may results in additional problems for their body. Our goal is to find the best treatment regimens that minimizes the cancer cell count and the deleterious effect of the drugs for a given patint. We examine the optimal control setting analytically, and include Grunwald-Letnikov numerical solutions to illustrate the optimal regimens under various assumptions.

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* Speaker.

1. Introduction

Chronic myelogenous leukemia (CML) is a kind of blood cancer, and Fokas and Adimy have presented CML models in 1991 and 2005, respectively. Recently, use of the models for analyzing the cancer reaction against drug therapy could assist physicians in cancer treatment. Therefore, using optimized control methods, which minimized damages to body, the drug dose can be optimized. In this article, at first we will introduce a fractional diffrential equations (FDE) model to present the interaction between naive T-cells, effectors T-cells, and CML cancer cells in cancer dormancy. Our goal in this work is to minimize the cancer cell population and the detrimental effects of the two types of drugs to the body of a hypothetical individual. We discuss the model with controls and present the objective functional, state the necessary conditions for the optimal control pair and charactrize optimal control pair in terms of the solution of the optimality system, which determind by using Grunwald-Letinkov method.

2. The CML Model

The model that we consider here is a three cells population model describing the interaction between the cancer cell population (C), the naive T-cell population (T_n) and effector T-cell population (T_e) [1]. We assume that the effector T-cells are specific to CML, activated by the presence of CML antigen and if we suppose these three cells evolve with independent variable time, then we can present our model in the form of FDE as follows;

$$\begin{aligned} D_t^\alpha T_n &= s_n - u_2(t) d_n T_n - k_n T_n \left(\frac{C}{C+\eta}\right); \\ D_t^\alpha T_e &= \alpha_n k_n T_n \left(\frac{C}{C+\eta}\right) + \alpha_e T_e \left(\frac{C}{C+\eta}\right) - u_2(t) d_e T_e - \gamma_e C T_e; \\ D_t^\alpha C &= (1-u_1(t)) r_c C ln \left(\frac{C_{max}}{C}\right) - u_2(t) d_c C - \gamma_c C T_e. \end{aligned} \tag{2.1}$$

In this system $T_n(0)$, $T_e(0)$ and $C(0)$ are known initial values and time dependent drug efficacies which incorporated by $u_1(t)$ and $u_2(t)$. All of the parameter values in the above equations are assumed to be positive. Among the several discretization methods for D_t^α, we use the one generated by Grunwald-Letnikov [2]. In this method $D^\alpha x(t)$ is approximated by

$$D^{\alpha}x(t) = \lim_{l\to\infty} l^{-\alpha}\sum_{j=0}^{[\frac{t}{l}]}(-1)^{j}\binom{\alpha}{j}x(t-jl),$$

L is step size and [t]is the integer part of t. Using this method for system (2.1), $D^{\alpha}x(t)$ is replaced by $\sum_{j=0}^{[\frac{t_n}{l}]} C_j^{\alpha}x(t_{n-j})$, where $t_n = nl$ and C_j^{α} is Grunwald-Letnikov coefficients defined by

$$C_0^{\alpha} = l^{-\alpha} \qquad C_j^{\alpha} = (1-\frac{1+\alpha}{j})c_{j-1}^{\alpha} \qquad j = 0,1,2,....$$

Now, system (2.1) can be discretize as follows

$$(T_n)_n = \frac{s_n - \sum_{j=1}^{n} c_j^{\alpha}(T_n)_{n-j}}{c_0 + d_n u_2(t) + k_n(\frac{c_n}{c_n+\eta})},$$

$$(T_e)_n = \frac{a_n k_n (T_n)_n(\frac{c_n}{c_n+\eta}) - \sum_{j=1}^{n} c_j^{\alpha}(T_e)_{n-j}}{c_0 + d_e u_2(t) + \gamma_e C_n - a_e(\frac{c_n}{c_n+\eta})},$$

$$(C)_n = \frac{-\sum_{j=1}^{n} c_j^{\alpha}(C)_{n-j}}{C_0 - (1-u_1(t)r_c ln(\frac{c_{max}}{c_n}) + d_c u_2(t) + \gamma_c(T_e)_n}.$$

3. Optimal Control Solution

Now we characterize the optimal control pair $(u_1{}^*, u_2{}^*)$ which gives the optimal drug dosage The existence of an optimal control pair is guaranteed by the compactness of the control and state spaces and the convexity of the problem. By considering the following objective functional;

$$Minimize \quad j(u_1,u_2) = \int_0^{t_f} C(t) + \frac{B_1}{2}u_1(t) + \frac{B_2}{2}u_2(t))\mathrm{d}t.$$

We apply the Pontryagin Maximum [3] Principle FDE from to obtain the following optimality conditions; [4]

$$D_t^{\alpha}\varphi_1(t) = -\frac{\partial H}{\partial T_n} = \varphi_1 u_2 d_n + \varphi_1 k_n(\frac{C}{C+\eta}) - \varphi_2\alpha_n k_n(\frac{C}{C+\eta});$$

$$D_t^{\alpha}\varphi_2(t) = -\frac{\partial H}{\partial T_e} = -\varphi_2\alpha_e(\frac{C}{C+\eta}) + \varphi_2 u_2 d_e + \varphi_2\lambda_e C + \varphi_3\lambda_c C;$$

$$D_t^{\alpha}\varphi_3(t) = -\frac{\partial H}{\partial C} = \varphi_1 k_n T_n\frac{\eta}{(C+\eta)^2} - (\varphi_2\eta(C+\eta))^2(\alpha_e T_e + \alpha_n k_n T_n) + \varphi_2\lambda_e T_e -$$

$$\varphi_3((1-u_1)r_c(ln\frac{C_{max}}{C} - 1) - u_2 d_c - T_e\lambda_c - 1);$$

$$D_t^\alpha T_n(t) = -\frac{\partial H}{\partial \varphi_1} = -s_n + u_2(t) d_n T_n + k_n T_n \left(\frac{C}{C+\eta}\right);$$

$$D_t^\alpha T_e(t) = -\frac{\partial H}{\partial \varphi_2} = -\alpha_n k_n T_n \left(\frac{C}{C+\eta}\right) - \alpha_e T_e \left(\frac{C}{C+\eta}\right) + u_2(t) d_e T_e + \gamma_e C T_e;$$

$$D_t^\alpha C(t) = -\frac{\partial H}{\partial \varphi_3} = -(1-u_1(t)) r_c C ln \left(\frac{C_{max}}{C}\right) + u_2(t) d_c C + \gamma_c C T_e.$$

$$\varphi_i(t_f) = 0, \qquad i = 1, 2, 3.$$

where the Hamiltonian H is calculated as;

$$H = C(t) + \frac{B_1}{2} u_1(t) + \frac{B_2}{2} u_2(t)) + \varphi_1 (D_t^\alpha T_n) + \varphi_2 (D_t^\alpha T_n) + \varphi_3 (D_t^\alpha C).$$

The Hamiltonian must be maximized with respect to the controls at the optimal control pair; thus by regarding $m_1 < u_1(t) < M_1$ and $m_2 < u_2(t) < M_2$, we would have:

$$u_1{}^* = \frac{\varphi_3 r_c C ln(\frac{C_{max}}{C})}{B_1}; \quad u_2{}^* = \frac{\varphi_1 d_n T_n + \varphi_2 d_e T_e + \varphi_3 d_c C}{B_2}.$$

Now we are able to use Grunwald-Letnikov method for discretzating the optimal control program and the numerical results could be carried out by using matlab software. The numerical simulation for a patient determined by TABLE 1 will be demonstraited in oral presentation.

TABLE 1. Patient features

s_n	d_n	d_e	d_c	k_n	η	α_n	α_e	C_{max}	r_c	λ_e	λ_c
0.29	0.35	.40	0.012	0.066	140	0.39	0.65	160000	0.011	0.079	0.058

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